MATHEMATICAL TRIPOS    Part III

Friday 30 May 2008   9.00 to 12.00

# PAPER 7

# COMMUTATIVE ALGEBRA

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

*In the following questions, A always denotes a commutative ring with unit element. All rings are tacitly assumed to be commutative and possess a unit element. Results presented in the lectures can be used without proof - unless you are explicitly asked to give a proof - but their use should be properly indicated. Results from examples sheets should not be used without proof.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) Let $\mathfrak{a} \subset A$ be an ideal and $\mathfrak{m} \subset A$ a prime ideal. We consider the quotient $B = A/\mathfrak{a}$ as an $A$-module in the obvious way. Show that the localisation $B_{\mathfrak{m}}$ of $B$ is the zero module if and only if $\mathfrak{a} \not\subset \mathfrak{m}$.

(b) Let $M$ be an $A$-module. Show that $M$ is the zero module if and only if for all maximal ideals $\mathfrak{m} \subset A$ the localisation $M_{\mathfrak{m}}$ is the zero module. (It may be helpful to consider a submodule $Ax \subset M$ generated by one element, and apply part (a).)

(c) Let $f : M \to N$ be an $A$-module homomorphism. Show that $f$ is injective if and only if for all maximal ideals $\mathfrak{m} \subset A$ the induced homomorphism $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ between the localisations is injective.

(d) Let $E$ be an $A$-module. Show that $E$ is flat if and only if for all maximal ideals $\mathfrak{m} \subset A$ the localisation $E_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module.

**2** (a) Give an example of a ring $B$, a $B$-module $N$ and an exact sequence of $B$-modules

$$0 \to M' \to M \to M'' \to 0$$

such that the induced sequence

$$0 \to \mathrm{Hom}_B(N, M') \to \mathrm{Hom}_B(N, M) \to \mathrm{Hom}_B(N, M'') \to 0$$

is not exact.

(b) Let $F$ be a flat $A$-module, and $P, Q \subset F$ be two submodules such that $F = P \oplus Q$. Show that $P$ is a flat $A$-module.

(c) An $A$-module $P$ is *projective* if for every exact sequence

$$0 \to M' \to M \to M'' \to 0$$

the induced sequence

$$0 \to \mathrm{Hom}_B(P, M') \to \mathrm{Hom}_B(P, M) \to \mathrm{Hom}_B(P, M'') \to 0$$

is exact. Show that a projective $A$-module is a direct summand of a free $A$-module. (You may first want to show that every module is the quotient of a free module.)

(d) Let $P$ be a projective $A$-module. Show that $P$ is flat.

**3** In the following $(A, \mathfrak{m})$ is a local Noetherian ring with residue field $k = A/\mathfrak{m}$. We put $d = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

(a) Show that $\mathfrak{m}$ can be generated by $d$ elements. In particular, the least number of generators of an $\mathfrak{m}$-primary ideal is less or equal to $d$.

(b) Prove that the ring of formal power series $B = A[[x]]$ is a local ring with maximal ideal $I = \mathfrak{m}B + xB$ and residue field $k$. Show that $\dim_k(I/I^2) = d + 1$ and $\dim(B) \leq d+1$. (When quoting a result from the lectures you may assume without proof that $B$ is Noetherian.)

(c) Now assume that $A$ is a regular local ring, i.e. $d = \dim(A)$. Let $B = A[[x]]$ be as above. Prove that $\dim(B) \geq d+1$. (One may consider a chain of prime ideals in $A$, and then construct a suitable chain of prime ideals in $B$.) Hence conclude that $\dim(B) = d+1$.

**4** (a) Let $\phi : A \to B$ be a ring homomorphism and $\mathfrak{p} \subset A$ be a prime ideal of $A$. Show that the set of prime ideals $\mathfrak{q}$ of $B$ with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ is in canonical bijection with $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, where $\kappa(\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$.

(b) Let $K$ be a field and $C$ a $K$-algebra which is finite-dimensional as $K$-vector space. Proof that every prime ideal of $C$ is maximal and that $C$ has only finitely many maximal ideals.

(c) Let $A \subset B$ be an integral ring extension, and suppose $B$ is finitely generated as $A$-algebra. Show that for every prime ideal $\mathfrak{p} \subset A$ there are only finitely many prime ideals $\mathfrak{q}$ of $B$ with $A \cap \mathfrak{q} = \mathfrak{p}$

## END OF PAPER