MATHEMATICAL TRIPOS Part III

Thursday 5 June 2008 9.00 to 11.00

PAPER 14

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{A} \subset \mathcal{P}[n]$ be an antichain. Prove that $\sum_{r=0}^{n} |\mathcal{A} \cap [n]^{(r)}| / \binom{n}{r} \leq 1$.

Let $a, x_1, \ldots, x_n \in \mathbb{R}^k$ satisfy $||x_i|| \geq 1$, $1 \leq i \leq n$. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the 2^n sums $\sum_1^n \epsilon_i x_i$, $\epsilon_i \in \{-1, 1\}$, lie in the open ball with centre a and radius 1.

Suppose now that $k = 1$. What is the greatest number of these sums that can lie in the open ball with centre a and radius 2?

2 A family $\mathcal{A} \subset \mathcal{P}[n]$ is *t-intersecting* if $|A \cap B| \geq t$ whenever $A, B \in \mathcal{A}$. State and prove an upper bound on the size of a t-intersecting family, and verify that it can be attained. (You may assume the Erdős-Ko-Rado theorem.)

Does every maximal 1-intersecting family attain the bound (for $t = 1$)? Does every maximal 2-intersecting family attain the bound (for $t = 2$)?

3 Define the *Shannon capacity* $c(G)$ of a graph G, and show that $c(G^2) = c(G)^2$.

Define an *orthonormal representation* of G, and the *Lovász θ-function* $\theta(G)$.

Prove that $c(G) \leq \theta(G)$, and deduce that $c(C_5) = \sqrt{5}$.

4 Prove that the Shannon capacity $c(G \sqcup \overline{G})$ of the disjoint union of the graph G and its complement satisfies $c(G \sqcup \overline{G}) \geq \sqrt{2n}$, where $n = |G|$.

Define a *representation* of G over a space M of polynomials, and prove that, if G has such a representation, then $c(G) \leq \dim M$.

Define a graph G such that $c(G \sqcup \overline{G}) > c(G) + c(\overline{G})$, and justify your claim.

END OF PAPER