UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 1.30 to 3.30

PAPER 12

COMBINATORICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove the Local LYM inequality, and deduce the LYM inequality. Which antichains in $\mathcal{P}([n])$ have size exactly $\binom{n}{\lfloor n/2 \rfloor}$?

(ii) A *symmetric chain* in $\mathcal{P}([n])$ is a chain $A_1 \subset \ldots \subset A_k$ $(k \geqslant 1)$ such that $|A_{i+1}| = |A_i| + 1$ for each $1 \leqslant i \leqslant k-1$ and also $|A_1| + |A_k| = n$. Prove that $\mathcal{P}([n])$ may be partitioned into symmetric chains. [*Hint: induction on n.*]

2 State the vertex-isoperimetric inequality in the discrete cube (Harper's theorem). Explain carefully how the Kruskal-Katona theorem may be deduced from Harper's theorem.

State the Erdős-Ko-Rado theorem, and give two proofs: one using the Kruskal-Katona theorem and one using cyclic orderings.

Which of the following are always true, for every n and every $r \leqslant n/2$, and which can be false? Give proofs or counterexamples as appropriate.

(i) If $\mathcal{A} \subset [n]^{(r)}$ is an intersecting family then the initial segment of the lexicographic order on $[n]^{(r)}$ of size $|\mathcal{A}|$ is also intersecting.

(ii) If $\mathcal{A} \subset [n]^{(r)}$ is an intersecting family then the initial segment of the colexicographic order on $[n]^{(r)}$ of size $|\mathcal{A}|$ is also intersecting.

(iii) If $\mathcal{A} \subset [n]^{(r)}$ is a 2-intersecting family then the initial segment of the lexicographic order on $[n]^{(r)}$ of size $|\mathcal{A}|$ is also 2-intersecting.

3 State and prove the edge-isoperimetric inequality in the discrete cube (the theorem of Harper, Lindsey, Bernstein and Hart).

Deduce that the isoperimetric number of the discrete cube is 1.

Which subsets of size 2^{n-1} of the discrete cube Q_n have edge-boundary of size exactly 2^{n-1}? Justify your answer.

4 State and prove the Frankl-Wilson theorem (on modular intersections).

Let p be prime. Using the Frankl-Wilson theorem, show that if $A \subset [4p]^{(2p)}$ satisfies $|x \cap y| \neq p$ for all $x, y \in A$ then $|A| \leqslant 2\binom{4p}{p-1}$.

Explain the Kahn-Kalai counterexample to Borsuk's conjecture.

Give, with justification, an explicit n such that Borsuk's conjecture is false in dimension n.

END OF PAPER