UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 9.00 to 11.00

PAPER 13

COMBINATORIAL PROBABILITY

*Attempt **THREE** questions,*

*At most **ONE** from any one of the **THREE** parts.*

*There are **SIX** questions in total.*

The questions carry equal weight.

The results you quote should always be stated precisely.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

FIRST PART

1 (i) Given $\Delta \geq 1$, write $\gamma(\Delta)$ for the maximal number c such that if $\mathcal{A} = \{A_1, \ldots, A_m\}$ is a family of events with a strong independence graph of maximal degree Δ, and $\mathbb{P}(A_i) < c$ for every i, then $\mathbb{P}(\bigcap_{i=1}^m \overline{A}_i) > 0$. Prove that $\gamma(1) = 1/2$ and

$$\gamma(\Delta) \leq (\Delta - 1)^{\Delta-1}/\Delta^{\Delta}$$

for $\Delta \geq 2$.

(ii) Let $\mathcal{A} = \{A_1, A_2, A_3\}$ be a family of events with $0 < \mathbb{P}(A_i) = a < 1$ for every i. Suppose that the oriented triangle is an independence digraph of $\mathcal{A}$. For what values of a can you guarantee that we have $\mathbb{P}(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3) > 0$? And what is the answer if instead the (unoriented) triangle is an independence graph of $\mathcal{A}$?

2 Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph (i.e., $\mathcal{E} \subset V^{(k)}$) such that every edge $E \in \mathcal{E}$ meets at most m other edges. A 3-*colouring* of H is a partition of the vertex set V into three sets, V_1, V_2, V_3, such that every edge meets each V_i. Define a correlation graph of a suitable family of events, and use it to prove that if $e(2m + 2) \leq 3^k$ then H has a 3-colouring.

SECOND PART

3 Let $f : Q_n \to \{0,1\}$ be a Boolean function with $\mathbb{P}(f = 1) = t$, and let β_i be the influence of the ith variable on f.

(i) Suppose that $\beta_i \leq \beta$ for every i. Show that if $\beta > 0$ is small enough and n is large enough then

$$\sum_{i=1}^{n} \beta_i \geq \frac{2}{3}\, t(1-t)\, \log(1/\beta). \tag{1}$$

(ii) Deduce that if n is large enough then

$$\max_i \beta_i \geq \frac{1}{2} t(1-t)(\log n)/n.$$

Hint for Part (i). Suppose, for a contradiction, that inequality (1) is false. Set $b = \frac{1}{3}\log(1/\beta)$. Show that the Fourier coefficients α_A of f satisfy

$$\sum_{1 \leq |A| \leq b} \alpha_A^2 \geq \frac{1}{2}\, t(1-t).$$

Deduce that for $\delta = 1/e$ we have

$$\sum_{i=1}^{n} \beta_i^{2/(1+\delta)} \geq 2\, b\, \delta^b\, t(1-t).$$

4 (i) Prove the Friedgut–Kalai theorem about sharp thresholds.

(You may assume a result about the influence of a variable in a weighted cube, provided it is stated precisely.)

(ii) Use the 'tribes' example to show that, apart frmo the constant, the Friedgut–Kalai theorem is best possible.

THIRD PART

5 (i) Consider (independent) bond percolation on $\mathbb{Z}^2$ with bond probability $1/2$. Write $h(m, n)$ for the probability that an m by n rectangle contains an open crossing from left to right. Sketch a proof of the assertion that $h(n + 1, n) = 1/2$ for every n.

(ii) Prove that $h(3n, 2n) \geq 2^{-7}$, and deduce that for every $\lambda > 0$ there is a constant $c_\lambda > 0$ such that $h(m, n) \geq c_\lambda$ whenever $m \leq \lambda n$.

(iii) Deduce Harris's theorem.

6 (i) State Harris's lemma, and deduce the nth root trick.

(ii) Consider (independent) bond percolation on $\mathbb{Z}^2$ with bond probability p. Write $h_p(m, n)$ for the probability that an m by n rectangle contains an open crossing from left to right. Assuming that $h_{1/2}(4n, n) > c_4$ for some constant $c_4 > 0$ and every n, use the $5n$ by $5n$ torus $\mathbb{T}_{5n}$ to prove that, given p, λ and ε with $1/2 < p < 1$, $\lambda > 0$, and $\varepsilon > 0$, we have $h_p(m, n) > 1 - \varepsilon$, provided n is large enough and $m \leq \lambda n$.

(iii) Show that there is a $p_0 < 1$ such that if $\widetilde{\mathbb{P}}$ is a 1-independent probability measure on $E(\mathbb{Z}^2)$ in which every bond is open with probability at least p_0 then $\widetilde{\mathbb{P}}(|C_0| = \infty) > 0$.

(iv) Prove Kesten's theorem.

END OF PAPER