UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2001 9 to 12

PAPER 76

COMBINATORIAL NUMBER THEORY

Attempt any ***THREE*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove Weyl's inequality. Explain briefly how it may be used to prove the following proposition: for every $\epsilon > 0$ there exists a positive integer N such that for every real number a there exists a positive integer $m \leq N$ such that am^2 is within ϵ of an integer. [You need *not* give technical details for the second part - all that is expected is a broad outline of the proof.]

2 Write an essay on Vinogradov's three-primes theorem. You should explain the general scheme of the proof and include at least some of the technical detail.

3 (i) State and prove Plunnecke's inequality. (You may assume Menger's theorem.)

(ii) Show how Plunnecke's inequality implies the following statement: if A is a set of integers and $|A + A| \leq C|A|$, then $|rA - sA| \leq C^{r+s}|A|$ for every pair of positive integers (r, s).

4 (i) State and prove the Balog–Szemerédi theorem.

(ii) Let A be a set of integers such that $|A| = n$ and $|kA - kA| \leq Cn$. Show that A has a large subset which is isomorphic to a subset of $\mathbb{Z}_N$, for some N which is not too large. (You should first make this statement precise, then prove it.)

(iii) Let $c > 0$, let n be sufficiently large and let A be a set of integers of size n containing at least cn^2 arithmetic progressions of length three. Assuming Szemerédi's theorem for progressions of length four, prove that A contains an arithmetic progression of length four. [*Hint: this follows from (i), (ii) and some easy Fourier analysis.*]

5 (i) Given a function f from $\mathbb{Z}_N$ to the unit disc in $\mathbb{C}$, or a subset of A of $\mathbb{Z}_N$, what does it mean to say that f or A is α-quadratically uniform?

(ii) Show that if A is a subset of $\mathbb{Z}_N$ of size δN, $\alpha > 0$ is sufficiently small and A is α-quadratically uniform, then the number of quadruples $(x, x + d, x + 2d, x + 3d)$ lying in A^4 is at least $\delta^4 N^2/2$. (Here all addition is mod N.)

(iii) Suppose that f is a function from $\mathbb{Z}_N$ to the unit disc, and let $B \subset \mathbb{Z}_N$ and $\phi : B \to \mathbb{Z}_N$ be such that

$$\sum_{\kappa \in B} |\Delta(f;k)^\wedge(\phi(\kappa))|^2 \geq \alpha N^3.$$

Prove that there are at least $\alpha^4 N^3$ quadruples $(a, b, c, d) \in B^4$ such that $a+b = c+d$ and $\phi(a) + \phi(b) = \phi(c) + \phi(d)$.