UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 8 June 2004 9 to 12

PAPER 80

COHOMOLOGY OF COHERENT SHEAVES

There are **five** *questions in total.*

Answer questions 1 and 2 and any two others.
In answering one question, you may take for granted any result which might be a part of a reasonable answer to a previous question.

Throughout this paper, k will be a fixed algebraically closed field and all varieties will be defined over k.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that if X is a smooth curve and Y is a projective variety, then every rational map from X to Y is in fact a morphism.

Explain the relation between finitely generated field extensions K/k of transcendence degree 1 and smooth projective curves over k.

Show that every non-constant morphism $C \to D$ of smooth projective curves is finite.

[*You may assume that the normalization of an integral domain A that is finitely generated over k in a finite extension of its field of fractions is finite over A.*] [20 marks]

2 Show that, for a coherent sheaf $\mathcal{F}$ on an affine variety X, $H^i(X, \mathcal{F}) = 0$ when $i \geq 1$.

Show that if U, V are affine open subvarieties of a variety X, then $U \cap V$ is also affine.

Derive from this a description of $H^i(X, \mathcal{F})$ for projective X, in terms of a covering by X of affine open subvarieties U_α and the modules $\Gamma(U_\alpha, \mathcal{F})$. [20 marks]

3 Show that for any smooth projective curve C, there are affine open subvarieties U_0, U_1 of C such that $C = U_0 \cup U_1$. Deduce that if $i \geq 2$, then $H^i(C, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on C and compute $\dim H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ for $i = 0, 1$ and all n. [30 marks]

4 Suppose that C is a smooth projective curve and that $\mathcal{F}$ is a coherent sheaf on C. Show that $\dim H^i(C, \mathcal{F})$ is finite for all i and that for all divisors D of sufficiently high degree, $H^i(C, \mathcal{F}(D)) = 0$. [30 marks]

5 Suppose that C is a smooth projective curve and that $\mathcal{M}$ is an invertible sheaf on C. Show that if $H^0(C, \mathcal{M}^\vee \otimes \Omega^1_C) = 0$, then $H^1(C, \mathcal{M}) = 0$. Deduce that there is a canonical isomorphism $H^1(C, \Omega^1_C) \cong k$. [30 marks]