UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2003 1.30 to 4.30

PAPER 23

CLASS FIELD THEORY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Write an essay on the cyclotomic field $K = \mathbb{Q}(\zeta_n)$. You should include a proof of the Artin reciprocity law for the extension $K/\mathbb{Q}$.

(ii) State the Kronecker-Weber theorem, and use a ramification argument to prove it in the case of quadratic fields.

2 (i) Let K be a finite extension of $\mathbb{Q}_p$. State and prove a version of Hensel's lemma, and use it to classify the unramified extensions of K.

(ii) Give a brief summary of the properties of the Hilbert norm residue symbol. Use Hensel's lemma to compute the groups $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ for all rational primes p. Give an explicit description of the quadratic Hilbert norm residue symbol in each case.

3 Write an essay on the Herbrand quotient and its application to norm index computations. Give full details either in the case of local fields or in the case of number fields. You should end by outlining the proof of the Hasse norm theorem.

4 Write an essay on central simple algebras and the Brauer group. You should include the construction of cyclic algebras and briefly indicate the role they play in computing the Brauer groups of finite fields, local fields and number fields.

5 (i) Let L/K be a Galois extension of number fields. Let E/K be a finite extension. Show that if $\mathfrak{p}$ is unramified in L/K then the primes above $\mathfrak{p}$ are unramified in LE/E.

(ii) Define the ray class group and the ray class field. Explain how to compute the order of the ray class group. Compute the ray class field for $K = \mathbb{Q}(\sqrt{-3})$ and $\mathfrak{m} = (7)$, and for $K = \mathbb{Q}(\sqrt{-6})$ and $\mathfrak{m} = (\sqrt{-6})$. Explain all your working.