UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 14 June, 2005 9 to 12

PAPER 26

CATEGORY THEORY

*You should attempt **ONE** question from Section A*

*and **TWO** from Section B.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Define the terms *monad*, *Eilenberg–Moore category* and *monadic functor*. State and prove the Precise Monadicity Theorem, and use it to show that the forgetful functor $\mathbf{KHaus} \to \mathbf{Set}$ is monadic, where $\mathbf{KHaus}$ denotes the category of compact Hausdorff spaces and continuous maps between them. [You may assume standard results from general topology, in particular the fact that a quotient X/R of a compact Hausdorff space X is Hausdorff if and only if the equivalence relation R is closed in $X \times X$.]

2 'Category theory is the one area of mathematics where definitions matter more than theorems'. Write a short essay arguing the case **either** for **or** against this statement.

SECTION B

3 State and prove the Yoneda Lemma.

A category $\mathcal{C}$ is called *cartesian closed* if it has finite products and, for every object A of $\mathcal{C}$, the functor $(-) \times A \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint (commonly denoted $(-)^A$). Show that the functor category $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed for any small category $\mathcal{C}$. [Hint: the Yoneda Lemma tells you how to define G^F, for objects F and G of $[\mathcal{C}, \mathbf{Set}]$; you may assume the result that any object of $[\mathcal{C}, \mathbf{Set}]$ is expressible as a colimit of representable functors.]

Show also that if $\mathcal{C}$ is a small cartesian closed category, then the Yoneda embedding $Y \colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ preserves exponentials, in the sense that we have a natural isomorphism $Y(B^A) \cong YB^{YA}$ for objects A, B of $\mathcal{C}$.

4 Explain what it means for an adjunction to be *idempotent*, and show that this condition is self-dual.

A *frame* is defined to be a complete lattice A in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all $a \in A$ and $S \subseteq A$; a *frame homomorphism* is a map preserving finite infima and arbitrary suprema. We write **Frm** for the category of frames and frame homomorphisms. Show that, for any topological space X, the lattice $\mathcal{O}(X)$ of open subsets of X is a frame, and that $X \mapsto \mathcal{O}(X)$ is a functor $\mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$.

We define a *point* of a frame A to be a frame homomorphism $A \to \{0,1\}$. Show that we may define a topology on the set $\mathcal{P}(A)$ of all points of A by taking the open sets to be those of the form

$$\{p \in \mathcal{P}(A) \mid p(a) = 1\}$$

for some $a \in A$. Show also that $\mathcal{P}$ is a functor $\mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$, and that it is right adjoint to $\mathcal{O}$. Is the adjunction $(\mathcal{O} \dashv \mathcal{P})$ idempotent?

5 Explain what it means for a category $\mathcal{C}$ to be *filtered*, and for $\mathcal{C}$ to be *weakly filtered*.

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called *discrete* if, given any $A \in \operatorname{ob} \mathcal{C}$ and any $f \colon FA \to B$ in $\mathcal{D}$, there exists a unique morphism $\widetilde{f}$ in $\mathcal{C}$ satisfying $\operatorname{dom} \widetilde{f} = A$ and $F(\widetilde{f}) = f$. If F is such a functor and $\mathcal{D}$ is weakly filtered, show that $\mathcal{C}$ is weakly filtered.

Let **Disc** denote the category of small categories and discrete functors between them, and let $\pi_0 : \mathbf{Disc} \to \mathbf{Set}$ denote the functor which sends a category to its set of connected components. Show that π_0 preserves pullbacks, provided the vertices of the pullback square are weakly filtered categories. [You may assume the result that the inclusion functor $\mathbf{Disc} \to \mathbf{Cat}$ creates pullbacks.]

Deduce that if $\mathcal{C}$ is a small filtered category then the functor $\mathbf{Disc}/\mathcal{C} \to \mathbf{Set}$ which sends $(F \colon \mathcal{D} \to \mathcal{C})$ to $\pi_0(\mathcal{D})$ preserves all finite limits.

6 Define the notions of *additive category* and *abelian category*. Prove that finite products and coproducts coincide in any additive category, and that any morphism in an abelian category can be factored, uniquely up to isomorphism, as an epimorphism followed by a monomorphism.

Give, with justification, an example of an additive category with finite limits and colimits, in which every epimorphism is a cokernel but not every monomorphism is a kernel. Show also that, in any such example, the class of normal monomorphisms (that is, monomorphisms which occur as kernels) must fail to be closed under composition.

END OF PAPER