UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 9 to 12

PAPER 69

BLACK HOLES

*Attempt **Question 1** and **THREE** other questions.*

The questions are of equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay describing why you believe black holes of mass M, charge Q and angular momentum J are described by a thermodynamic temperature T with

$$T = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2})} \ .$$

2 Describe *briefly* how to prove that the mass of a spacetime vanishes if there is a solution ϵ to the Killing spinor equation

$$\nabla_a \epsilon = 0 \ .$$

Suppose that general relativity is modified so that the analogue of Killing spinors now obey the equation

$$D_a \epsilon \equiv \nabla_a \epsilon + c\gamma_a \epsilon = 0 \ ,$$

where c is a real positive constant. By considering

$$D_{[a} D_{b]} \epsilon = 0 \ ,$$

show that

$$\left(R_{abcd}\gamma^{cd} + 16c^2 \gamma_{ab}\right)\epsilon = 0 \ ,$$

where

$$\gamma_{ab} = \tfrac{1}{2}[\gamma_a, \gamma_b] \ .$$

Now assume that the space of solutions to this equation has four complex dimensions, and that the Ricci tensor has rank four (that is, it is not degenerate), prove that the Ricci tensor obeys the Einstein equation

$$R_{ab} = -24c^2 g_{ab} \ .$$

3 In a variant of general relativity, the metric of a spherically symmetric spacetime containing a central object of mass M and charge Q is, in Schwarzschild-like co-ordinates,

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r(r - Q^2/M)(d\theta^2 + \sin^2\theta d\phi^2)\ .$$

What are the conditions on M and Q for this metric to represent

(i) a black hole

(ii) a naked singularity

[You may assume that there is a curvature singularity at $r = Q^2/M$.]

Assuming that light travels along null geodesics in this theory, calculate the absorption cross-section for light by the central object described by this metric in the two cases $Q^2 = 0$ and $Q^2 = 2M^2$.

4

The Reissner-Nordstrom spacetime has metric

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ .$$

Either by considering the Euclidean version of this spacetime, or by calculating the surface gravity of the black hole directly, derive an expression for the temperature of the hole. State clearly any assumptions you make in your calculations.

Suppose a black hole has initial mass M_0, and charge Q, with $M_0 \gg |Q|$. The hole can be assumed to radiate charged particles of mass m only if its temperature T is greater than m, but not otherwise. Thus, if the temperature $T < m$, Q is constant in the evaporation process. Sketch a graph of T versus M (for fixed Q). Assuming that T is never greater than m, what is the final state of this system?

Find the condition on m and Q for this to occur.

5

Sketch a proof of the area theorem for black holes by deriving the formula

$$\frac{d^2}{d\lambda^2}A^{1/2} = \left(-\tfrac{1}{2}R_{ab}p^a p^b + \sigma^2\right)A^{1/2} \ ,$$

where $p^a = dx^a/d\lambda$ represents the four-vector tangent to the world-line of photons that are moving along the outermost trapped surfaces of area A, λ is an affine parameter for these geodesics and the shear σ is defined by

$$\sigma^2 = \tfrac{1}{2}(\nabla_a p_b)(\nabla^a p^b) - \tfrac{1}{4}(\nabla_a p^a)(\nabla_b p^b) \ .$$

The energy-momentum tensor T_{ab} of a perfect fluid of pressure p, energy density ρ, velocity vector u^a is

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab} \ .$$

What are the conditions on p and ρ that guarantee that the black hole area theorem must hold?

Suppose that the cosmological constant Λ were non-vanishing. Would this affect your result?