UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 1.30 to 4.30

## PAPER 79

## ASYMPTOTIC METHODS IN FLUID MECHANICS

*Attempt **THREE** questions.*

*The questions carry equation weight.*

*There are **FIVE** questions in total.*

*The maximum credit available for question 1 will be divided equally between part (a) and part (b).*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) (i) Evaluate the first two terms as $m \nearrow 1$ of the elliptic integral

$$\int_0^{\pi/2} \frac{1}{(1-m^2\sin^2\theta)^{1/2}}\,\mathrm{d}\theta\,,$$

given that

$$\int \frac{1}{(1+u^2)^{1/2}}\,\mathrm{d}u = \log\left(u+(1+u^2)^{1/2}\right)$$

and

$$\int \frac{1}{\sin\varphi}\,\mathrm{d}\varphi = \log\left(\tan\tfrac{1}{2}\varphi\right)\,.$$

(ii) Evaluate the first two terms as $x \to \infty$ of the integral

$$\int_0^{\infty} \frac{e^{-xu}}{(\log u)^2 + e^{-2xu}}\,\mathrm{d}u\,.$$

(b) (i) Briefly describe the use of the Briggs-Bers technique to determine the long-time behaviour of solutions of initial-value problems, and in particular state how to determine whether or not a system is absolutely unstable by considering its dispersion relation.

Consider the third-order equation

$$\frac{\partial\eta}{\partial t} + \alpha\frac{\partial^2\eta}{\partial x^2} + \beta\frac{\partial^3\eta}{\partial x^3} + \gamma\eta = 0\,,$$

where $\alpha,\beta,\gamma$ are complex constants. By considering solutions proportional to $\exp(-\mathrm{i}\omega t + \mathrm{i}kx)$, derive conditions on $\alpha$ and $\beta$ for the system to possess a finite maximum temporal growth rate over all real $k$, and explain why this is important for application of the Briggs-Bers technique. Determine necessary conditions for the occurrence of absolute instability.

(ii) The nonlinear Schrodinger (NLS) equation in one dimension is

$$-\mathrm{i}\frac{\partial A}{\partial t} + \chi\frac{\partial^2 A}{\partial x^2} - \mu A + \delta A|A|^2 = 0\,,$$

where $\chi,\delta$ and $\mu$ are real positive constants. We investigate the stability of the constant solution of NLS by writing

$$A = \left\{\left(\frac{\mu}{\delta}\right)^{1/2} + \epsilon A_1\right\}\exp(\mathrm{i}\epsilon\theta)\,,$$

where $A_1 = A_1(x,t)$ and $\theta = \theta(x,t)$ are unknown real functions and $\epsilon \ll 1$. By taking terms of size $O(\epsilon)$, find a system of coupled equations for $A_1$ and $\theta$. Assuming that

$$\theta, A_1 \propto \exp(-\mathrm{i}\Omega t + \mathrm{i}Kx)\,,$$

determine the dispersion relationship between $\Omega$ and $K$. Comment on the stability of the constant solution of NLS.

**2** By means of Watson's lemma, show that the asymptotic expansion of the integral

$$F(z) = \int_1^\infty \exp(-z^3 t^3)\, \mathrm{d}t$$

for real $z \gg 1$, is

$$F(z) \sim \frac{\exp(-z^3)}{3z^3} \sum_{r=0}^{\infty} \frac{\Gamma(r+\frac{2}{3})}{\Gamma(\frac{2}{3})} \frac{1}{(-z)^{3r}} \,.$$

Hence find an asymptotic expansion, including exponentially small terms, for

$$G(z) = \int_0^1 \exp(-z^3 t^3)\, \mathrm{d}t \,.$$

Now suppose that $z$ is complex. By considering contours of steepest descent, find the asymptotic expansions of $G(z)$ for $0 \le \arg(z) \le 2\pi$ away from Stokes lines, which should be identified. Selected contours of constant $\Re(-z^3t^3)/|z^3|$ for complex $z$ are plotted in figure 1 (see overleaf), together with contours of $\Im(-z^3t^3)/|z^3|$ passing through $t = 0$ and $t = 1$.

Obtain an expression for the smoothing out of the jump in the sub-dominant term at one of Stokes lines; confirm that your result is consistent with the asymptotic expansions away from that Stokes lines obtained earlier.

*You may quote the following results.*

(a) The Borel sum of

$$\sum_{p=0}^{\infty} \frac{\Gamma(\gamma+p)e^\lambda}{\lambda^{p+\gamma}} \,,$$

for real $\gamma \ge 0$ and $\Re(\lambda) > 0$, is

$$I(\lambda,\gamma) = \int_0^\infty \frac{t^{\gamma-1}\exp(\lambda(1-t))}{1-t}\, \mathrm{d}t \,,$$

where the contour of integration is assumed to pass just above the pole at $t = 1$.

(b) If $\gamma \gg 1$ and

$$\lambda \sim \gamma + \mathrm{i}\mu\gamma^{\frac{1}{2}} + \nu + \dots \,,$$

where $\mu = O(1)$ and $\nu = O(1)$ then

$$I(\lambda,\gamma) \sim i\pi \left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)$$

(c)

$$\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \tfrac{2}{\sqrt{3}}\pi \,.$$

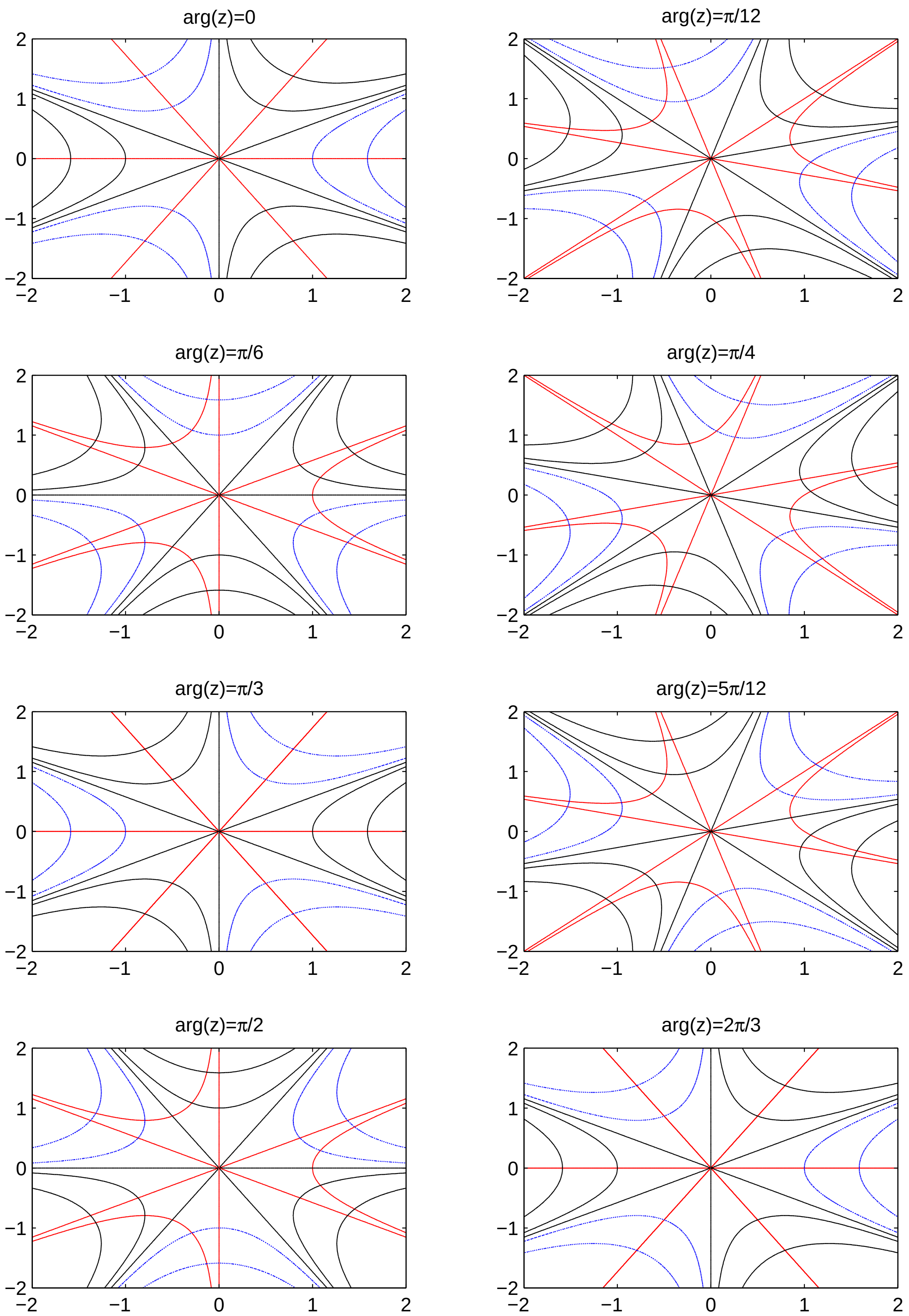


Figure 1. Contours in the complex $t$-plane of constant $\Re(-z^3t^3)/|z^3|$ for given complex $z$ (black or blue depending on whether the real part is positive or strictly negative), plus contours of $\Im(-z^3t^3)/|z^3|$ passing through $t=0$ and $t=1$ (red).

**3** The function $y(x)$ satisfies the nonlinear differential equation

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y^{\frac{1}{2}} + \varepsilon k = 0\,,$$

where $k > 0$ is an order one constant, subject to the boundary conditions $y(0) = 0$ and $y(1) = 1$. In the limit $\varepsilon \to 0$, use the method of matched asymptotic expansions to find the leading-order solution for $0 \leq x \leq 1$. An implicit solution is acceptable as long as is it demonstrated that the solution satisfies any necessary matching conditions. Sketch the solution.

*Hints.*

(a) Away from the boundaries there is an outer region. Near $x = 0$ there is a single inner region, while near $x = 1$ there are two inner asymptotic regions (with the smallest region lying *between* the outer region and the asymptotic inner region that includes $x = 1$).

(b) The implicit solution to

$$\tfrac{1}{2}\left(\frac{\mathrm{d}Y}{\mathrm{d}\xi}\right)^2 - \tfrac{2}{3}Y^{\frac{3}{2}} + kY = \tfrac{1}{3}k^3$$

is given by

$$\sigma(\xi - \xi_0) = \sqrt{2k}\log\left|\frac{\sqrt{k + 2Y^{\frac{1}{2}}} - \sqrt{3k}}{\sqrt{k + 2Y^{\frac{1}{2}}} + \sqrt{3k}}\right| + \sqrt{6(k + 2Y^{\frac{1}{2}})}\,,$$

where $\xi_0$ is a constant and $\sigma = \pm 1$, with the sign being chosen as needed and/or appropriate. Note that

(i) if $|Y^{\frac{1}{2}} - k| \ll 1$, then $\sigma\xi \ll -1$ and

$$\ln|Y^{\frac{1}{2}} - k| \sim \frac{\sigma\xi}{\sqrt{2k}} + \ldots\,;$$

(ii) if $Y^{\frac{1}{2}} \gg 1$, then $\sigma\xi \gg 1$ and

$$Y^{\frac{1}{2}} \sim \tfrac{1}{12}\xi^2 + \tfrac{1}{6}\xi\xi_0 + \ldots\,.$$

(c) For $z > 0$ define

$$\zeta = \frac{\sqrt{3}}{2}\int_1^z \frac{\mathrm{d}u}{\sqrt{u^{\frac{3}{2}} + c}}\,,$$

where $c$ is a constant. If $c = 0$, then

$$z^{\frac{1}{2}} = \tfrac{1}{12}\left(\zeta - \zeta_0\right)^2 \qquad \text{where} \qquad \zeta_0 = -2\sqrt{3}\,.$$

while if $c \neq 0$ then when $z \ll 1$

$$z \sim 2\sqrt{\frac{c}{3}}\,(\zeta - \zeta_0) \qquad \text{where} \qquad \zeta_0 = -\frac{\sqrt{3}}{2}\int_0^1 \frac{\mathrm{d}u}{\sqrt{u^{\frac{3}{2}} + c}}\,.$$

**4** (a) Consider the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \epsilon f(x) \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0 \,,$$

subject to $x = 1$ and $\mathrm{d}x/\mathrm{d}t = 0$ when $t = 0$, where $\epsilon \ll 1$ and $f(x)$ is a given function. Use the method of multiple scales to find the leading-order approximation to $x(t;\epsilon)$ which is uniformly valid for $t \leq O(1/\epsilon)$ when

(i) $f(x) = x^2 - 1$,

(ii) $f(x) = \sin x$,

(iii) $f(x) = |x|$.

[Hint: in (iii) you may find it useful to write

$$|\sin\tau| \cos\tau = \sum_{n=0}^{\infty} a_n \cos(n\tau)$$

for some constants $a_n$.]

(b) Consider the linearised Ginsburg-Landau equation

$$\frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} - \mu\eta - \gamma \frac{\partial^2 \eta}{\partial x^2} = 0$$

with $\mu(X) = \mu_0 - \nu X^2$ and other coefficients constant, where $X = \epsilon x$, $\epsilon \ll 1$ and we take $\nu > 0$. A global mode is of the form $A = a(x)\exp(-i\omega_G t)$, with $a(x)$ bounded as $x \to \pm\infty$. Show, by writing $a(x) = \exp(Ux/2\gamma)b(\xi)$, where $\xi = \sqrt{\epsilon}cx$ for suitable $c$, that the global mode satisfies

$$\frac{d^2 b}{d\xi^2} + (\lambda - \xi^2)b = 0 \,, \qquad (*)$$

for suitable $\lambda$, to be found. By seeking a solution of $(*)$ in the form $\exp(-\xi^2/2)H_n(\xi)$, where $H_n$ is a power series, show that the solutions of $(^*)$ grow exponentially as $\xi \to \pm\infty$, unless $\lambda = (2n+1)$ for any positive integer $n$. Hence, determine the allowed global mode frequencies and comment on the global stability of the system.

**5** (a) Consider the motion of a fluid along a straight channel aligned parallel to the $x$-axis, with walls $y = \pm 1$ and which is infinite in the spanwise $z$ direction. The mean flow along the channel is $U(y)$ in the $x$ direction, and if we assume that unsteady disturbances to the $y$ component of velocity take the form $v(y) \exp(-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}lz)$, then it follows that $v(y)$ satisfies the Orr-Sommerfeld equation

$$(U - c)\left(\frac{d^2}{dy^2} - k^2 - l^2\right) v - \frac{d^2U}{dy^2} v - \frac{1}{\mathrm{i}kR}\left(\frac{d^2}{dy^2} - k^2 - l^2\right)^2 v = 0 \,, \qquad (*)$$

where $c = \omega/k$ and $R$ is the Reynolds number. The boundary conditions are that $v$ and $dv/dy$ vanish on $y = \pm 1$.

(i) In the inviscid case $R = \infty$, prove Rayleigh's Theorem that a necessary condition for the existence of temporal instability is that $U(y)$ possesses an inflection point in the channel.

(ii) Show that the solution of the Orr-Sommerfeld equation (*) for given values of $k, l, R$ can be related to the solution of the same equation but with $l = 0$ and with $k, R$ replaced by new values $k', R'$ (to be determined). Use this to explain why the Orr-Sommerfeld equation first becomes unstable to two-dimensional disturbances as the Reynolds number is increased.

(iii) The Squire equation is

$$(-\mathrm{i}\omega + \mathrm{i}kU)\eta - \frac{1}{R}\left(\frac{d^2}{dy^2} - k^2 - l^2\right)\eta = 0 \,,$$

subject to $\eta(\pm 1) = 0$. Show that solutions of the Squire equation are temporally stable. [Hint: Multiply the Squire equation by the complex conjugate of $\eta(y)$ and integrate over $-1 \leq y \leq 1$.]

(b) Consider the first-order ordinary differential equation

$$\frac{d\mathbf{q}}{dt} = A\mathbf{q} \,,$$

where $\mathbf{q}(t)$ is an $N$-dimensional vector and $A$ is a constant $N \times N$ matrix. The optimal growth, $G(t)$, is defined to be the maximum value of

$$\frac{\parallel \mathbf{q}(t) \parallel}{\parallel \mathbf{q}(0) \parallel}$$

maximised over all $\mathbf{q}(0) \neq 0$. Determine lower and upper bounds on $G(t)$ in terms of properties of the matrix $A$, and calculate these bounds explicitly in the case

$$M = \begin{pmatrix} -\frac{1}{R} & 0 \\ 1 & -\frac{2}{R} \end{pmatrix} . \qquad (**)$$

For the choice of $M$ given in (**), show that

$$\exp(Mt) = \begin{pmatrix} \exp(-t/R) & 0 \\ -[\exp(-2t/R) - \exp(-t/R)]R & \exp(-2t/R) \end{pmatrix} .$$

Determine the initial conditions which lead to optimal growth at time $t$ when (i) $t \gg R$, (ii)$t \ll R$.

## END OF PAPER