UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 9.00 to 12.00

## PAPER 76

## ASYMPTOTIC METHODS IN FLUID DYNAMICS

*Candidates may attempt* **ALL** *questions.*

*There are* **five** *questions in total.*

*The questions are of equal weight.*

*A distinction mark may be obtained by good answers to approximately* **THREE** *questions.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) The integral $J(\lambda;\varepsilon)$ is defined for real $\lambda > 0$ by

$$J(\lambda;\varepsilon) = \int_0^1 \frac{1}{(\lambda(x-1)+1)^2 + \varepsilon^2\cos^2(\pi x/2)}\,\mathrm{d}x\,.$$

If $0 < \varepsilon \ll 1$ calculate the leading-order asymptotic approximation of $J(\lambda;\varepsilon)$ for $0 < \lambda < 1$, $\lambda$ close to 1 (where how close to one should be specified), and $\lambda > 1$. Briefly discuss whether the approximation for $\lambda > 1$ is uniformly valid for $\lambda \gg 1$.

(b) In the following, the Fourier Transform is defined to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \exp(-\mathrm{i}kx) f(x)\mathrm{d}x\ .$$

(i) Show, being careful to explain each step in your reasoning, that the Fourier Transform of the Heaviside step function is

$$\pi\delta(k) + \frac{\mathrm{i}}{k}\ .$$

Hence, find the Fourier Transform of $1/x$.

(ii) Show, by using your answer to (i), or otherwise, that the Fourier Transform of $\log|x|$ is

$$-\frac{\pi}{|k|} + C\delta(k)\ ,$$

where $C$ is a constant which need not be determined.

(iii) Using your answers to (i) & (ii), and quoting without proof any standard results concerning Fourier Transforms that you require, solve for $f(x)$ the integral equation

$$⨍_{-\infty}^{\infty} \frac{f(t)}{x-t}\mathrm{d}t = \mathrm{H}(x-1) - \mathrm{H}(x+1)\ .$$

(iv) Determine the first **two** terms in the large-wavenumber expansion of the Fourier Transform of the modified Bessel function $\mathrm{K}_0(|x|)$.

[*Hint: Note that* $\mathrm{K}_0(z)$ *is the solution of the equation*

$$z^2\frac{\mathrm{d}^2\mathrm{K}_0}{\mathrm{d}z^2} + z\frac{\mathrm{dK}_0}{\mathrm{d}z} - z^2K_0 = 0$$

*such that* $\mathrm{K}_0(z) = -f(z)\log(z) + g(z)$, *where* $f$ *and* $g$ *are analytic functions, with* $f(0) = 1$, *and such that* $\mathrm{K}_0(|z|) \to 0$ *as* $|z| \to \infty$. ]

**2**

(i) The integral $I(\lambda, n)$ is defined for real integer $n$ and $Re(\lambda) > 0$ by

$$I(\lambda, n) = \int_0^\infty \frac{t^{n-1}\exp(\lambda(1-t))}{1-t}\,\mathrm{d}t\,,$$

where the contour of integration is assumed to pass just above the pole at $t = 1$. Suppose that $n \gg 1$ and that

$$\lambda \sim n + \mathrm{i}\mu n^{\frac{1}{2}} + \nu + \ldots\,,$$

where $\mu = O(1)$ and $\nu = O(1)$. By considering $\frac{\partial I}{\partial \mu}$ or otherwise, find the leading-order asymptotic expansion for $I(\lambda, n)$ as $n \to \infty$.

(ii) Show that, formally,

$$I(\lambda, n) \sim \exp(\lambda) \sum_{r=n}^{\infty} \frac{\Gamma(r)}{\lambda^r}\,.$$

(iii) The asymptotic series for the Airy function, $\mathrm{Ai}(z)$, when $|z| \gg 1$ and $|\arg(z)| < \pi$ is given by

$$\mathrm{Ai}(z) = \frac{1}{2z^{\frac{1}{4}}\pi^{\frac{1}{2}}} \exp\left(\tfrac{1}{2}\sigma\right) \sum_{r=1}^{\infty} Y_r\,,$$

where

$$Y_r = \frac{\Gamma(r+\frac{1}{6})\Gamma(r+\frac{5}{6})}{2\pi\sigma^r\Gamma(r+1)} \quad \text{and} \quad \sigma = -\tfrac{4}{3} z^{\frac{3}{2}}\,.$$

Show that

$$Y_r \to \frac{\Gamma(r)}{2\pi\sigma^r} \quad \text{as} \quad r \to \infty\,.$$

*Comment.* You may quote Stirling's asymptotic approximation for the Gamma function

$$(r-1)! = \Gamma(r) \sim \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} r^r\,\mathrm{e}^{-r} \quad \text{as} \quad r \to \infty\,.$$

(iv) Hence show that if the asymptotic series for the Airy function is truncated at the smallest term, say $r = n$, then for

$$\arg(\sigma) = \frac{\phi}{|\sigma|^{\frac{1}{2}}} \quad \text{where} \quad \phi = O(1)\,,$$

the remainder $R_n$ is given by

$$R_n \sim \frac{\mathrm{i}\exp\left(-\frac{1}{2}\sigma\right)}{4z^{\frac{1}{4}}\pi^{\frac{1}{2}}} \left(1 + \mathrm{erf}\left(\frac{\phi}{2^{\frac{1}{2}}}\right)\right)$$

Briefly interpret this result.

**3** The function $y(x)$ satisfies the differential equation

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + xy + y^2 = 0\,.$$

If $y(a) = \alpha$ and $y(1) = \beta$, with $0 < a < 1$, identify for what values of $\alpha$ and $\beta$ solutions can be found in the limit $\varepsilon \to 0$ on the assumption that there are no rapid oscillations or 'internal' boundary layers away from the end points. Sketch your solution[s] indicating whether there is a unique solution. Briefly discuss whether additional solutions with internal boundary layers can be easily ruled out.

Next suppose that $a = \alpha = 0$, i.e. $y(0) = 0$. Derive the governing equation in the inner region near $x = 0$ and state the boundary conditions a solution should satisfy. Without solving the equation exactly, discuss whether a solution satisfying the boundary conditions is likely to exist, e.g. on the basis of linearising the equation in the 'intermediate matching region' and discussing the linearised equation's solutions.

*Comment.* You may quote the result that

$$\int \frac{\mathrm{d}y}{\sqrt{a^3 - 3ay^2 - 2y^3}} = -\frac{2}{\sqrt{3a}} \operatorname{arctanh}\left(\sqrt{\frac{a-2y}{3a}}\right)\,.$$

**4**

(a) (i) Consider the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \epsilon c(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + x = k\ ,$$

subject to $x = \mathrm{d}x/\mathrm{d}t = 0$ when $t = 0$, where $c$, $k$ and $\epsilon$ are real positive constants with $\epsilon \ll 1$. Use the method of multiple scales to find the leading-order approximation to $x(t;\epsilon)$ which is uniformly valid for $t \leq O(1/\epsilon)$.

(ii) Now consider the coupled system

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \epsilon y(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + x = k$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \epsilon(1+x-y)$$

such that $x = \mathrm{d}x/\mathrm{d}t = y = 0$ when $t = 0$. Find the uniformly-valid leading-order approximations for $x(t;\epsilon)$ and $y(t;\epsilon)$ when $t \leq O(1/\epsilon)$.

(b) The non-dimensional equations describing the inviscid one-dimensional motion of a gas are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + M\frac{\partial(\rho u)}{\partial x} &= 0 \\ (1+M\rho)\left(\frac{\partial u}{\partial t} + Mu\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} &= 0 \qquad (1) \\ (1+M\gamma p) &= (1+M\rho)^\gamma\ , \end{aligned}$$

where $p$, $\rho$ and $u$ are the unsteady pressure and density perturbation and the velocity respectively, $\gamma$ is a positive constant and $M \ll 1$ is the constant Mach number. By writing

$$p(x,t;M) = p_0(\theta, X) + Mp_1(\theta, X) + ....\ ,$$

with similar expansions for $p$ and $u$, where $X = Mx$ and $\theta = t - x$, show that $p_0$ satisfies

$$\frac{\partial p_0}{\partial X} - \left(\frac{\gamma+1}{2}\right)p_0\frac{\partial p_0}{\partial \theta} = 0\ , \qquad (2)$$

uniformly in $|\theta| \leq O(M^{-1})$.

How does equation (2) change if the viscous term $\beta\partial^2u/\partial x^2$, with $\beta = O(M)$, is introduced onto the right-hand side of equation (1)?

**5**

(a) (i) Describe the use of the Briggs-Bers technique to determine the long-time behaviour of solutions of the one-dimensional linearised Ginzburg-Landau equation with constant real coefficients,

$$\mathcal{L}\eta \equiv \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} - \mu\eta - \gamma\frac{\partial^2 \eta}{\partial x^2} = 0 \,.$$

Find the conditions involving $U, \mu$ and $\gamma$ for the occurrence of convective instability and of absolute instability.

(ii) By taking Fourier transforms in both $x$ and $t$, find the solution of

$$\mathcal{L}\eta = \delta(x)\delta(t) \,.$$

In the limit $t \to \infty$ relate the behaviour of your solution to the conditions found in (i) above.

(iii) In two dimensions the linearised Ginzburg-Landau equation with constant coefficients is

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + V\frac{\partial \eta}{\partial y} - \mu\eta - \gamma_{11}\frac{\partial^2 \eta}{\partial x^2} - \gamma_{22}\frac{\partial^2 \eta}{\partial y^2} - 2\gamma_{12}\frac{\partial^2 \eta}{\partial x \partial y} = 0 \,,$$

where $U, V, \mu$ and $\gamma_{ij}$ are real constants. By considering solutions of the form $\eta = \exp(-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly)$ and computing the group velocity, $\left(\frac{\partial \omega}{\partial k}\,,\ \frac{\partial \omega}{\partial l}\right)$, find a necessary condition for the occurrence of absolute instability in the form

$$\mu > f(U, V) \,,$$

where $f(U, V)$ is a quadratic function of $U, V$ to be determined.

Given that the vector $(U, V)$ represents a mean-flow velocity whose magnitude is fixed but whose orientation can be changed, find the minimum value of $\mu$ required to guarantee the absence of absolute instability for **any** mean-flow direction.

(b) Consider the linearised Ginzburg-Landau equation in one dimension on the semi-infinite domain $x > 0$, with boundary conditions $A = 0$ at $x = 0$ and $A \to 0$ as $x \to \infty$. The coefficients $U$ and $\gamma$ are real constants, but $\mu = \mu_0 - \epsilon\lambda x$ with $\epsilon \ll 1$, and $\mu_0$ and $\lambda$ positive real constants.

By considering a solution of the form

$$f(\epsilon^\sigma x)\exp\left(\frac{Ux}{2\gamma} - \mathrm{i}\omega t\right) \,,$$

where the index $\sigma > 0$ and the function $f$ are to be determined, show that the global mode frequencies are

$$\mathrm{i}\left(\mu_0 - \frac{U^2}{4\gamma} + (\epsilon^2\gamma\lambda^2)^{\frac{1}{3}} z_n\right) \qquad\qquad n = 1, 2, 3, \ldots\ldots \,,$$

where $z_n$ is the $n$th zero of the Airy function $\mathrm{Ai}(x)$ on the negative real axis. Hence, deduce the condition for global instability, and compare this to the condition for local absolute instability.

[*Hint:* $\mathrm{Ai}(x)$ *is the solution of* $y'' = xy$ *which is bounded as* $x \to \infty$. $\mathrm{Ai}$ *has a countable number of real zeros, all negative.* ]