UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday 27 May, 2004 1.30 to 3.30

## PAPER 64

## ASTROPHYSICAL FLUID DYNAMICS

*Attempt* **THREE** *questions.*

*There are* **four** *questions in total.*

*The questions carry equal weight.*

*Candidates may bring their notebooks into the examination. The following equations may be assumed.*

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho \nabla \Phi + \mathbf{j} \wedge \mathbf{B}$$

$$\rho \frac{De}{Dt} = \frac{p}{\rho} \frac{D\rho}{Dt} + \operatorname{div} (\lambda \nabla T) + \epsilon$$

$$\operatorname{div} \mathbf{B} = 0; \, \mathbf{j} = \mu_0^{-1} \operatorname{curl} \mathbf{B}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl} (\mathbf{u} \wedge \mathbf{B})$$

$$p = (\gamma - 1)\rho e = \frac{\mathcal{R}}{\mu} \rho T$$

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** A fluid flow with radial velocity $u(r)$ and density $\rho(r)$ represents the steady, spherically symmetric accretion of an isentropic fluid from a surrounding medium of uniform density $\rho_\infty$ onto a gravitating point mass $M$ centred at the origin $(r = 0)$. Show that $u(r)$ and the adiabatic sound speed $c(r)$ obey the equation

$$\frac{1}{u}\frac{du}{dr} = \frac{1}{r} \cdot \frac{GM/r - 2c^2}{c^2 - u^2}$$

and find the corresponding equation for $dc/dr$.

Show that at the radius $r$ at which the flow is trans-sonic, the velocity is given by $u^2 = GM/2r$.

Verify that at large radii the equations permit a solution of the form $u \to 0$, $\rho \to \rho_\infty$ as $r \to \infty$.

Verify that at small radii the equations permit a solution of the form $u^2 \sim 2GM/r$ and $u^2 \gg c^2$ as $r \to 0$, provided that the ratio of specific heat, $\gamma$ is such that $\gamma < \gamma_{crit}$, where the value of $\gamma_{crit}$ is to be determined.

A strongly magnetised star, radius $R_*$, has a dipole field of the form $\mathbf{B} = \nabla\psi$, where, using cylindrical polar coordinates $(R, \phi, z), \psi \propto z/\left(R^2 + z^2\right)^{3/2}$. Bondi-like accretion is taking place in an axisymmetric fashion along the magnetic field lines onto a small circle of radius $a\,(\ll R_*)$ at the magnetic pole. Show that the cross-sectional area of the flow is approximately $A = \pi a^2\,(r/R_*)^3$, for $A \ll R_*^2$.

If the magnetic field is so strong that it remains exactly dipolar, show that it exerts no force on the fluid.

In this case find the corresponding value of $\gamma_{crit}$ which allows supersonic flow onto the star.

**2** A static, gaseous plane-parallel atmosphere rests on a fixed base at $z = 0$ and is subject to a fixed gravitational field $\mathbf{g} = (0, 0, -g(z))$ where $g(z) = \Omega^2 z$, and $\Omega$ is a constant. The atmosphere is isothermal, with (isothermal) sound speed $c_s$. Show that the density structure is of the form

$$\rho(z) = \rho_0 \exp\left(-z^2/2H^2\right) \quad (0 \leqslant z < \infty),$$

where $\rho_0$ is the density at $z = 0$, and $H$ is a constant.

The atmosphere is subject to small adiabatic velocity perturbations of the form $\mathbf{u} = \left(0, 0, w(z)e^{i\sigma t}\right)$, where $\sigma$ is the oscillation frequency. What kind of modes would you expect to find in such an atmosphere and what kind of modes does this perturbation represent?

Show that $w(z)$ obeys the equation

$$\frac{d^2 w}{dz^2} - \frac{z}{H^2}\frac{dw}{dz} + \frac{1}{\gamma}\left(\frac{\sigma^2}{c_s^2} - \frac{1}{H^2}\right) w = 0\,,$$

where $\gamma$ is the usual ratio of specific heats.

By considering a series solution about $z = 0$, or otherwise, show that the oscillation modes $w_n(z)$ with finite kinetic energy $E = \int_0^\infty \frac{1}{2}\rho w^2 dz$ take the form of polynomials of degree $2n + 1 (n = 0, 1, 2, \ldots)$ and find the corresponding mode frequencies $\sigma_n$.

**3** A uniform density, incompressible fluid contains a bounded shear layer with velocity field, in Cartesian coordinates, of the form $\mathbf{U}_0 = (U(z), 0, 0)$, where

$$U(z) = \begin{cases} U_0 & z \geqslant d \\ (z/d)U_0 & |z| < d \\ -U_o & z \leqslant -d \end{cases}$$

It is subject to a perturbation of the form

$$\mathbf{u}(x, z, t) = (u(z), 0, w(z)) \exp[ik(x - ct)],$$

and

$$p'(x, z, t) = p'(z) \exp[ik(x - ct)],$$

where $k$ is real and $k > 0$. Obtain an equation for $w$ and show that an appropriate solution is of the form

$$w(z) = \begin{cases} A \exp[-k(z-d)] & z \geqslant d \\ B \exp[-k(z-d)] + C \exp[k(z+d)] & |z| < d \\ D \exp[k(z+d)] & z \leqslant -d\,. \end{cases}$$

where $A$, $B$, $C$and $D$ are constants.

Explain why the quantity

$$\left[(c - U)\frac{dw}{dz} + \frac{dU}{dz}w\right]$$

is continuous at $z = \pm d$.

Hence, or otherwise, show that

$$\left(\frac{c}{U_0}\right)^2 = \frac{1}{4\alpha^2}\left\{(2\alpha - 1)^2 - e^{-4\alpha}\right\},$$

where $\alpha = kd$.

Deduce that disturbances with wavelengths such that $0 < \alpha < \alpha_s$ are unstable for some $\alpha_s$ in the range $\left(\frac{1}{2} < \alpha_s < 1\right)$, and give a physical interpretation of this result.

**4** An incompressible fluid of uniform density $\rho$ is rotating uniformly with angular velocity $\Omega$ inside a rigid smooth sphere of radius $r_0$. Find the pressure distribution in the fluid.

The flow undergoes small oscillations such that, for example, the Eulerian pressure perturbation is of the form

$$p' = p'(R,z)\exp\left\{i\left(\sigma t + m\phi\right)\right\},$$

where $(R,\phi,z)$ are cylindrical polar coordinates, $m$ is the azimuthal wave number and $\sigma$ is the oscillation frequency. Show that $W \equiv p'/\rho$ satisfies the equation

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) - \frac{m^2 W}{R^2} + \left(1 - \frac{4\Omega^2}{\overline{\sigma}^2}\right)\frac{\partial^2 W}{dz^2} = 0,$$

where $\overline{\sigma} = \sigma + m\Omega$.

Derive an appropriate boundary condition for $W$ on the sphere $r = r_0$.

Show that there is an axisymmetric oscillation mode of the form

$$W = z\left(AR^2 + Bz^2 + C\right),$$

where $A, B$ and $C$ are constants, and find the corresponding oscillation frequency.

[**Note**: the equations of motion may be assumed to be

$$\frac{\partial u_R}{\partial t} + \mathbf{u}\cdot(\nabla u_R) - \frac{u_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R}$$

$$\frac{\partial u_\phi}{\partial t} + \mathbf{u}\cdot(\nabla u_\phi) + \frac{u_R u_\phi}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \phi}$$

$$\frac{\partial u_z}{\partial t} + \mathbf{u}\cdot(\nabla u_z) = -\frac{1}{\rho}\frac{\partial p}{\partial z}]$$