UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2008 1.30 to 3.30

PAPER 16

ARITHMETIC COMBINATORICS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove that if $\delta > 0$ and N is sufficiently large, then every subset $A \subset \{1, 2, \ldots, N\}$ of cardinality at least δN contains an arithmetic progression of length 3.

2 Let A be a subset of cardinality n of an Abelian group G. Prove the following assertions about A.

(i) Let $\alpha > 0$, and suppose that there are at least αn^3 quadruples $(x, y, z, w) \in A^4$ such that $x + y = z + w$. Then A has a subset B of cardinality at least cn such that the difference set $B - B$ has cardinality at most Cn, where c and C are positive constants that depend on α only.

(ii) If $A - A$ has cardinality at most Cn then $2A - 2A$ has cardinality at most Kn, where K is a constant that depends on C only.

(iii) If G is the group $\mathbb{Z}_N$, and if $n \geqslant \theta N$, then $2A - 2A$ contains an arithmetic progression of length aN^b, where a and b are positive constants that depend on θ only.

3 (i) Let G be an Abelian group and let $f : G \to \mathbb{C}$. Define the U^3-norm $\|f\|_{U^3}$ of f, and prove that it is a norm.

(ii) Let α be a positive constant. Suppose that $\|f_i\|_\infty \leqslant 1$ for $1 \leqslant i \leqslant 4$, and that $\|f_i\|_{U^3} \leqslant \alpha$ for at least one i. Prove that

$$|\mathbb{E}_{x,d}\, f_1(x) f_2(x + d) f_3(x + 2d) f_4(x + 3d)| \leqslant \beta,$$

where β is a positive constant that tends to zero as α tends to zero.

(iii) Let A be a subset of $\mathbb{Z}_N$ of cardinality at least δN, and let $\alpha > 0$. What does it mean to say that A is *quadratically α-uniform*? Prove that if α is sufficiently small and A is quadratically α-uniform, then A contains a "genuine" arithmetic progression of length 4: that is, a set of size 4 that continues to be an arithmetic progression when it is identified in the obvious way with a subset of $\{0, 1, \ldots, N - 1\}$.

END OF PAPER