UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 1.30 to 4.30

PAPER 67

APPROXIMATION THEORY

*Attempt **FOUR** questions.*

*There are **SIX** question in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For $f \in C[0,1]$, write down the definition of the Bernstein polynomial $B_n(f)$ and prove that, for any polynomial p of degree $m < n$, the Bernstein polynomial $B_n(p)$ is also of degree m.

Hence, or otherwise, show that, uniformly in $[0,1]$,

$$B_n(p) \to p \quad (n \to \infty), \quad \text{for} \quad p(x) = 1,\ x, \text{ or } x^2.$$

2 For a 2π-periodic function $f \in C(\mathbb{T})$, write down the definitions of the Fourier sum $s_n(f)$ and of the Fejer sum $\sigma_n(f)$ of degree n and $n-1$, respectively.

Consider the so-called de la Vallee Poussin sum

$$v_{n,m}(f) := \tfrac{1}{m}\Big(s_n(f) + s_{n+1}(f) + \cdots + s_{n+m-1}(f)\Big)\,.$$

(a) Show that, for any trigonometric polynomial t_n of degree n, we have

$$v_{n,m}(t_n) = t_n$$

for any m.

(b) Find an expression for $v_{n,m}$ in terms of two Fejer sums σ_k and σ_ℓ and use it to derive the bound

$$\|v_{n,m}(f)\|_\infty \le \left(\tfrac{2n}{m} + 1\right)\, \|f\|_\infty \quad \forall\, f \in C(\mathbb{T})\,.$$

(c) Let $\frac{n}{m} \le M$. Combine (a) and (b) to establish the inequality

$$\|f - v_{n,m}(f)\|_\infty \le 2(M+1)\, E_n(f) \quad \forall\, f \in C(\mathbb{T})\,,$$

where $E_n(f)$ is the best uniform approximation of f from $\mathcal{T}_n$, the space of all trigonometric polynomials of degree n.

3 Let σ_n be the Fejer operator, i.e., for a 2π-periodic function $f \in C(\mathbb{T})$,

$$\sigma_n(f,x) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)\,dt, \qquad F_n(t) := \tfrac{1}{2n}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}, \qquad \tfrac{1}{\pi}\int_{-\pi}^{\pi} F_n(t)dt = 1.$$

Prove that, for any $\alpha \in (0,1)$, the following estimate is valid:

$$\|\sigma_n(f) - f\|_\infty \le c_\alpha\, \omega\Big(f, \tfrac{1}{n^\alpha}\Big)\,,$$

where $\omega(f,\delta)$ is the (first) modulus of continuity of f, and c_α is a constant that depends only on α.

4 (a) State the Chebyshev alternation theorem for the element of best uniform approximation to a function $f \in C[-1,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n.

(b) Let $T_n(x) = \cos n \arccos x$ be the Chebyshev polynomial of degree n, and let

$$f(x) = \sum_{k=0}^{\infty} a_k T_{3^k}(x), \quad \text{where} \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty, \quad x \in [-1,1].$$

For every n, find p_n, the polynomial of best approximation to f in $C[-1,1]$, and compute the value of $E_n(f)$.

5 (I) Let $\mathcal{S}_k(\Delta)$ be the space of splines of degree $k-1$ spanned by the L_∞-normalized B-splines $(N_j)_{j=1}^n$, on a knot sequence $\Delta = (t_j)_{j=1}^{n+k}$, where $t_j < t_{j+k}$. Let $x = (x_i)_{i=1}^n$ be interpolation points obeying the conditions

$$N_i(x_i) > 0\,,$$

and let $P_x : C[a,b] \to \mathcal{S}_k(\Delta)$ be the map which, given any $f \in C[a,b]$, provides the spline $P_x(f)$ from $\mathcal{S}_k$ which interpolates f at (x_i). Prove that

$$\|P_x\|_{L_\infty} \le \|A_x^{-1}\|_{\ell_\infty}$$

where A_x is the matrix $(N_j(x_i))_{i,j=1}^n$.

(II) Consider the case of cubic interpolating splines on the uniform knot-sequence $(t_1, t_2, \dots, t_{n+4}) = (1, 2, \dots, n+4)$ with the interpolating points

$$x_i = \tfrac{1}{2}(t_i + t_{i+4}) = i + 2, \quad i = 1, \dots, n\,.$$

(a) Using the recurrence relation between B-splines, or otherwise, determine the values of N_j at the points (x_i).

(b) Write down the matrix $A_x = (N_j(x_i))$, and evaluate the norm $\|A^{-1}\|_{\ell_\infty}$. [*You may use any appropriate theorem on the inverse of certain matrices if correctly stated*].

(c) Hence show that $\|P_x\|_{L_\infty} \le 3$.

6 For a knot sequence $\Delta = (t_i)_{i=1}^{n+k} \subset [a,b]$ with distinct knots, let

$$M_i(t) := k\,[t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}\,, \quad N_i(t) := (t_{i+k} - t_i)\,[t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the sequences of L_1- and L_∞-normalized B-splines, respectively. Prove that

$$\text{(a)} \quad M_0(t) = k \sum_{i=0}^{k} \frac{(t_i - t)_+^{k-1}}{\omega'(t_i)}, \qquad \text{where} \quad \omega(x) = \prod_{i=0}^{k} (x - t_i)\,.$$

Prove also that

$$\text{(b)} \quad \int_a^b M_i(t)dt = 1, \qquad \text{(c)} \quad \sum_{i=1}^{n} N_i(t) = 1, \quad t_k \le t \le t_{n+1}.$$

END OF PAPER