UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 1.30 to 4.30

PAPER 69

APPROXIMATION THEORY

Attempt **FOUR** *questions.*

There are **seven** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) For $f \in C[0,1]$, write down the definition of the Bernstein polynomial $B_n(f)$ of degree n, and prove that $\|B_n(f)\|_\infty \le \|f\|_\infty$.

b) Let $f_{n0} \equiv 1$ and

$$f_{nm}(x) := x\left(x - \tfrac{1}{n}\right)\left(x - \tfrac{2}{n}\right)\cdots\left(x - \tfrac{m-1}{n}\right), \quad 1 \le m \le n\,.$$

Show that $B_n(f_{nm}, x) = f_{nm}(1)x^m$.

c) Using a) and b) prove that $B_n(g_m) \to g_m$ uniformly for any polynomial $g_m(x) = x^m$.

[*Hint*: Write $\|B_n(g_m) - g_m)\| \le \|B_n(g_m - f_{nm})\| + \|B_n(f_{nm}) - g_m\|$.]

2 1) For a 2π-periodic function $f \in C(\mathbb{T})$, write down the definitions of the Fourier sum $s_n(f)$ and of the Fejer sum $\sigma_n(f)$ of degree n.

2) Consider the so-called de la Vallee Poussin sum

$$v_n(f) := \tfrac{1}{n}\Big[s_n(f) + s_{n+1}(f) + \cdots + s_{2n-1}(f)\Big]\,.$$

a) Show that, for any trigonometric polynomial t_n of degree n, it follows that $v_n(t_n) = t_n$.

b) Find an expression for v_n in terms of two Fejer sums σ_m and σ_ℓ and use it to derive the bound

$$\|v_n(f)\|_\infty \le 3\,\|f\|_\infty \quad \forall\, f \in C(\mathbb{T})\,.$$

c) Combine a) and b) to establish the inequality

$$\|f - v_n(f)\|_\infty \le 4E_n(f) \quad \forall\, f \in C(\mathbb{T})\,,$$

where $E_n(f)$ is the best uniform approximation of f from $\mathcal{T}_n$, the space of all trigonometric polynomials of degree n.

3 a) Formulate the Kolmogorov criterion for the element of best approximation to a real-valued function $f \in C[0,1]$ from a linear subspace $\mathcal{A}$ of $C[0,1]$.

b) From this criterion, derive the Chebyshev alternation theorem for the element of best approximation to a function $f \in C[0,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n.

c) Show that, to each $f \in C[0,1]$, there corresponds a system of points $(x_{n,k})$ such that if $\ell_n(f)$ is the interpolating polynomial to f on nodes $x_{n,0}, x_{n,1}, \ldots, x_{n,n}$, then

$$\|\ell_n(f) - f\|_\infty \to 0.$$

4 Let σ_{n-1} be the Fejer operator, i.e., for a 2π-periodic function $f \in C(\mathbb{T})$,

$$\sigma_{n-1}(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)F_{n-1}(t)\,dt, \quad F_{n-1}(t) := \frac{1}{2n}\frac{\sin^2 \frac{n}{2}t}{\sin^2 \frac{1}{2}t}, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} F_{n-1}(t)dt = 1.$$

Prove that, if $f \in \operatorname{Lip}\alpha$, for some $0 < \alpha < 1$, i.e.

$$|f(x) - f(y)| \le M\,|x-y|^\alpha\,,$$

then

$$\|\sigma_{n-1}(f) - f\| \le \text{constant} \cdot n^{-\alpha}\,.$$

5 1) Formulate the Markov lemma about zeros of the k-th derivatives of polynomials $p, q \in \mathcal{P}_n$ whose zeros interlace.

2) For $\Delta = (t_i)_{i=0}^n$, a sequence of $n+1$ distinct nodes in $[-1, 1]$, set

$$M_k(x) := \sup\{|p^{(k)}(x)| : p \in \mathcal{P}_n,\ |p(t_i)| \le 1\}.$$

Further, let $p_* \in \mathcal{P}_n$ be the polynomial such that

$$p_*(t_i) = (-1)^i, \quad i = 0, ..., n,$$

and let $(q_s)_{s=1}^n \in \mathcal{P}_n$ be the polynomials such that

$$q_s(t_i) = \begin{cases} (-1)^i, & i < s, \\ (-1)^{i+1}, & i \ge s. \end{cases}$$

Prove that, for any $x \in [-1, 1]$,

$$\text{either} \quad M_k(x) = |p_*^{(k)}(x)|, \qquad \text{or} \quad M_k(x) = |q_s^{(k)}(x)| \quad \text{for some } s\,.$$

[*Hint*: Use the Lagrange interpolation formula with the Markov lemma applied to appropriate polynomials.]

6 For a knot sequence $\Delta = (t_i)_{i=1}^{n+k} \subset [a, b]$, let

$$M_i(t) := k\,[t_i, ..., t_{i+k}](\cdot - t)_+^{k-1}\,, \quad N_i(t) := (t_{i+k} - t_i)\,[t_i, ..., t_{i+k}](\cdot - t)_+^{k-1}$$

be the sequences of L_1- and L_∞-normalized B-splines, respectively. Prove that

$$a) \quad \int_a^b M_i(t)dt = 1, \qquad b) \quad \sum_{i=1}^n N_i(t) = 1, \quad t_k \le t \le t_{n+1}.$$

7 Let (N_i) and (M_i) be the B-spline bases of degree $k-1$ with L_∞- and L_1-normalization, respectively, defined on a knot sequence $\Delta = (t_i)_{i=1}^{n+k} \subset [0,1]$.

Given $f \in C[0,1]$, let

$$P_{\mathcal{S}}(f) := s^* = \sum_{j=1}^{n} a_j N_j$$

be the orthogonal projection of f onto $\mathcal{S} := \operatorname{span}(N_j)$ with respect to the ordinary inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$. Then $P_{\mathcal{S}}$ is also well defined as an operator from $C[0,1]$ onto $C[0,1]$.

a) Show that the max-norm of $P_{\mathcal{S}}$ satisfies the inequality

$$\|P_{\mathcal{S}}\|_\infty \le \|G^{-1}\|_{\ell_\infty}\,,$$

where $G = (g_{ij})$ is the Gram matrix with the elements $g_{ij} = (M_i, N_j)$.

b) For linear splines ($k=2$) and arbitrary Δ, compute the entries of the i-th row of G in terms of $h_\nu = t_{\nu+1} - t_\nu$.

c) Using the fact that G is totally positive, or otherwise, prove the estimate

$$\|G^{-1}\|_{\ell_\infty} \le 3\,, \qquad k = 2\,.$$

[*You may use any appropriate theorems on the inverse of certain matrices if correctly stated.*]