MATHEMATICAL TRIPOS Part III

Friday 6 June 2008 9.00 to 12.00

PAPER 65

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt **FOUR** *questions.*

There are **SEVEN** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write a brief account of exterior algebra, differential forms, Hodge duality and Stokes' theorem. Illustrate your answer by reference to the following exercise:

In $2+1$ dimensional Minkowski spacetime, the action for a one-form field A is

$$\frac{1}{2}\int\left(F\wedge\star F+kA\wedge F\right)$$

where $F=dA$, and k is a constant. Obtain the equations of motion in terms of the one-form $B=\star F$.

Explain why the equation of motion, but not the action, is invariant under the gauge transformation $A\to A+d\Lambda$.

2 Write a brief account of left and right translation on a Lie group. Hence, show that every Lie group admits two flat connections with torsion and a torsion-free connection. Show, by calculating the curvature and Ricci tensor, that the latter is in general not flat.

What special feature occurs if the Lie group is assumed to be semi-simple?

3 Define a (left) fibre bundle and a principal bundle. Show that every principal bundle admits a global right action of the structural group. Show that a principal bundle is trivial if and only if it admits a global section.

If G is a Lie group and H_1 and H_2 Lie subgroups, comment on the cases G/H_1,$H_2\backslash G$, and $H_2\backslash G/H_1=(H_2\backslash G)/H_1=H_2\backslash(G/H_1)$.

Define an associated bundle and illustrate your answers by reference to the orthonormal frame bundle and the tangent bundle of a manifold. Hence show that the frame bundle and tangent bundle of a Lie group are trivial.

4 You are given that a set of $\frac{1}{2}n(n-1)$ one-forms $\lambda_{ij}=-\lambda_{ji}$, $i,j=1,2,\ldots,n$ for $so(n)$ satisfy

$$d\lambda_{ij}=\lambda_{ik}\wedge\lambda_{kj}\,.$$

Show that the Jacobi identity is satisfied.

By considering the basis

$$(\lambda_{12}\pm\lambda_{43},\lambda_{23}\pm\lambda_{41},\lambda_{31}\pm\lambda_{42})\,,$$

for $so(4)$, show that

$$so(4)=so(3)\oplus so(3)\,.$$

5 Show, by establishing the isomorphism

$$SO(2,2) \equiv SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2\,,$$

that three dimensional anti-de-Sitter spacetime, AdS_3, may be identified with a suitable group manifold. Use your results to show that AdS_3 satisfies the Einstein equations with cosmological constant.

What can you say about AdS_3 modulo the antipodal map (i.e. if points $x \in AdS_3 \in \mathbb{E}^{2,2}$ are identified with $-x$)? What can be said if the global time coordinate is not identified?

6 Define a Poisson manifold and a symplectic manifold. Give examples of Poisson manifolds which cannot be thought of as symplectic manifolds. Give a condition that a Poisson manifold gives rise to an algebra satisfying the Jacobi identity, and show that this condition is satisfied in the case of a symplectic manifold.

In the semi-classical theory of an electron of charge e moving on a Fermi surface in a uniform magnetic field $\mathbf{B}$, the equation satisfied by the momentum vector $\mathbf{p} \in \mathbb{R}^3$ is

$$\frac{d\mathbf{p}}{dt} = e\mathbf{v} \times \mathbf{B}\,,$$

where

$$\mathbf{v} = \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}}\,,$$

for a so-called dispersion function $E(\mathbf{p})$.

Interpret this equation in terms of Poisson geometry, identifying the Poisson bi-vector. Is the Jacobi identity satisfied? Justify your answer.

7 Define a Lagrangian submanifold of a symplectic manifold. Indicate briefly the use of Lagrangian submanifolds for geometric quantisation.

The first law of thermodynamic reads

$$dM = TdS - PdV + \mu dN\,,$$

where M is a function of the extensive variables $(S, V, N) \in \mathbb{R}^3$ and the intensive variables $(T, -P, \mu) \in \mathbb{R}^3$ are regarded as functions of the extensive variables. Show that every choice of the function M defines, locally at least, a Lagrangian submanifold of $\mathbb{R}^6 = T^\star\mathbb{R}^3$.

END OF PAPER