UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 1.30 to 4.30

PAPER 64

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt **FOUR** questions.*

*There are **SEVEN** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain Cartan's procedure for obtaining the curvature of a metric using differential forms.

Hence, obtain the curvature of the metric

$$ds^2 = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2}$$

and show that the metric is an Einstein metric.

What can you say about the isometries of the metric?

2 Explain, using Stokes's theorem, how the equation

$$d \star J = 0$$

for a one-form J, gives rise to a conserved charge.

A certain theory of the Quantum Hall Effect in 2+1 dimensional Minkowski spacetime $\mathbb{E}^{2,1}$, is governed by the action

$$S = \int_{\mathbb{E}^{2,1}} \Big(A \wedge dA - c \star J \wedge A \Big),$$

where A and J are one-forms and c is a constant. Show that if the action is gauge invariant, then the current J must be conserved.

Obtain the equations of motion for A and show that they imply that the current J is conserved. If γ is a closed loop at constant time, obtain a relation between

$$\Phi = \int_\gamma A$$

and the total charge enclosed by γ.

If A is a $U(1)$ connection, what is the significance of Φ?

Calculate the energy momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$

3 Define the moment map $\mu : P \to \mathfrak{g}^\star$ for the action of a group G on a symplectic manifold $\{P, \omega\}$. Derive a condition on G ensuring that the Poisson algebra generated by the moment map coincides with the Lie algebra $\mathfrak{g}$ of G.

If $P = T^\star \mathbb{R}^3$ with its standard symplectic structure and the Hamiltonian is

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{Mm}{r},$$

show that the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

and Runge-Lenz vector

$$\mathbf{K} = \mathbf{p} \times \mathbf{L} - \frac{Mm\mathbf{r}}{r}$$

Poisson commute with H. Given the Poisson brackets

$$\{L_i, L_j\} = \epsilon_{ijk} L_k,$$

$$\{L_i, K_j\} = \epsilon_{ijk} K_k,$$

$$\{K_i, K_j\} = -2H\epsilon_{ijk} L_k,$$

what can you say about the action generated by $\mathbf{L}$ and $\mathbf{K}$ on the level sets of H?

4 Define a *principal fibre bundle* and an associated vector bundle. Under what circumstances is a principal bundle trivial?

What does it mean to say that a manifold is *parallelisable*? Give examples of spheres which are parallelisable.

Show that every group manifold is parallelisable.

Show that every semi-simple group admits an Einstein metric.

5 Establish the isomorphisms

(a) $SO(4) \equiv SU(2) \times SU(2)/\mathbb{Z}_2$,

(b) $SO(3,1) \equiv SL(2,\mathbb{C})/\mathbb{Z}_2$,

(c) $SO(2,2) \equiv SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$,

(d) $SO(3) \equiv SU(2)/\mathbb{Z}_2$,

(e) $SO(2,1) \equiv SL(2,\mathbb{R})/\mathbb{Z}_2$.

Show further that the unit quaternions can be identified with the three-sphere S^3.

By regarding

$$\omega \wedge \omega$$

as a quadratic form on $\omega \in \Lambda^2(\mathbb{R}^4)$, show that $SO(3,3) \equiv SL(4,\mathbb{R})/\mathbb{Z}_2$.

In all cases, where relevant, the identity component of the group should be taken.

6 Explain how, using the Marsden-Weinstein reduction procedure, starting with a $2n$-dimensional symplectic manifold $\{P, \omega\}$ and a Lie group G acting by symplecto-morphisms, one may obtain a new symplectic manifold $\{P', \omega'\}$ of dimension $2n-2g$, where $g = \dim G$.

Illustrate your description by considering either an isotropic simple harmonic oscillator or a free particle moving in the plane.

7 Write an essay on geometric quantisation. Your essay should include a treatment of pre-quantisation and the problem of finding a suitable polarisation.

END OF PAPER