UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 1.30 to 3.30

PAPER 71

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt **ALL THREE** questions. The questions are of equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain *briefly*, using an example of your choice, integration and Stokes' theorem using differential forms.

Two particles with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ separated by a distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ lie inside a ball of radius R, $r_1 = |\mathbf{r}_1| \leq R$, $r_2 = |\mathbf{r}_2| \leq R$ in three dimensional Euclidean space $\mathbb{E}^3$. If $x = r_1 + r_2$ and $y = r_1 - r_2$, show that $2R - |y| \geq x \geq r$ and $|y| \leq r$. Assuming that the position vectors are distributed uniformly inside the ball, show, using differential forms, that the probability dP of the separation r lying between r and $r + dr$ is given by

$$dP = \left(\frac{3r^2}{R^3} - \frac{9r^3}{4R^4} + \frac{3r^5}{16R^6} \right) dr.$$

Hint Start with

$$\left(\frac{3}{4\pi R^3} \right)^2 r_1^2 \sin\theta_1 dr_1 \wedge d\theta_1 \wedge d\phi_1 \wedge r^2 \sin\theta dr \wedge d\theta \wedge d\chi,$$

where (r_1, θ_1, ϕ_1) are the polar coordinates of the first particle with respect to the origin and (r, θ, χ) are polar angles of particle one with respect to particle two, θ being the angle between $\mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}_1$ and χ is the polar angle of particle two about the vector $\mathbf{r}_1$. Using the cosine formula for a triangle, eliminate $d\theta$ in favour of dr and integrate over $r_1, r_2, \theta_1, \phi, \chi$.

2 The euclidean group in two dimensions, E(2), may be represented by the set of matrices

$$\begin{pmatrix} \cos\psi & -\sin\psi & x \\ \sin\psi & \cos\psi & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Choosing an appropriate basis for the Lie algebra e(2), calculate the commutation relations.

If $\mathbf{L}_i, i = 1, 2, 3$ is a basis of left-invariant vector fields on $E(2)$, a one parameter family of connections ∇ on the tangent bundle $TE(2)$ is such that acting on a vector field Z,

$$\nabla_{\mathbf{L}_i} Z = \lambda \big[\mathbf{L}_i, Z \big],$$

for $\lambda \in \mathbb{R}$. Calculate the components of the Ricci tensor of the family of connections. Comment on the cases $\lambda = 0$ and $\lambda = 1$.

3 Define a symplectic manifold and explain how the functions on it may be endowed with a Poisson algebra structure. What is a Hamiltonian vector field? Relate the Lie Bracket of two Hamiltonian vector fields to the Poisson brackets of the associated moment maps.

N identical vortices moving on the plane have as phase space $(\mathbb{R}^2)^N$ endowed with symplectic form

$$\sum_{a=1}^{N} dx_a \wedge dy_a$$

and Hamiltonian

$$H(x_a, y_a) = -\sum_{a<b} \ln\big((x_a - x_b)^2 + (y_a - y_b)^2\big).$$

Obtain the equations of motion and indicate how this is compatible with the interpretation as vortex motion.

Show that the quantity

$$L(x_a, y_a) = \frac{1}{2}\sum_{a=1}^{N}(x_a^2 + y_a^2)$$

Poisson commutes with H. Show that L is the moment map associated to the standard action of $SO(2)$ on $(\mathbb{R}^2)^N$.

A system of such vortices rigidly rotates with angular velocity Ω. Show that if (x'_a, y'_a) are coordinates co-rotating with angular velocity Ω, so that (x'_a, y'_a) are independent of time, then they are critical points of the function

$$H(x'_a, y'_a) + \Omega L(x'_a, y'_a).$$