UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 6 June, 2005 9 to 12

PAPER 13

ALGEBRAIC TOPOLOGY

*Attempt **THREE** questions.*

*There are **FIVE** questios in total.*

The questions carry equal weight.

Throughout, you may assume (i) general algebraic facts about short and long exact sequences (ii) closed manifolds admit cell complex structures which make a chosen closed submanifold a subcomplex (iii) the computation of the (co)homology groups of spheres (iv) Eilenberg-Maclane spaces are unique up to homotopy equivalence.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What are the reduced homology groups $\tilde{H}_*(X;\mathbb{Z})$ of a topological space X, and what are the relative homology groups $H_*(X,A;\mathbb{Z})$ where $A \subset X$ is a subspace?

Give a careful statement of the Excision Theorem. Deduce $H_*(X,A;\mathbb{Z}) \cong \tilde{H}_*(X/A;\mathbb{Z})$ if X is a cell complex and $A \subset X$ is a subcomplex. Hence, or otherwise, show that if Σ_g is a fixed closed oriented surface of genus g and $A \subset \Sigma_g$ is an embedded circle then $H_*(\Sigma_g, A;\mathbb{Z})$ determines whether A is nullhomologous but not whether it is nullhomotopic.

If A is an embedded disjoint union of 2 circles, what are the possibilities for $H_*(\Sigma_g, A;\mathbb{Z})$? *Briefly* justify your answer.

2 Define the *cup-product* of cochains, and prove that this product descends to a well-defined product in cohomology.

Describe without proof the cohomology ring of real projective space with $\mathbb{Z}_2$ coefficients. Deduce that if $n > m$ then $f : \mathbb{RP}^n \to \mathbb{RP}^m$ induces the zero map in reduced cohomology.

By considering the expression $\frac{f(x)-f(-x)}{|f(x)-f(-x)|}$, or otherwise, show that given any $f : \mathbb{S}^n \to \mathbb{R}^n$ there is a point $x \in \mathbb{S}$ such that $f(x) = f(-x)$. Hence deduce that if $\mathbb{S}^n$ is written as a union of $(n+1)$ closed sets, at least one of the closed sets contains a pair of antipodal points.

Would the last fact remain true if we took 4 closed sets on $\mathbb{S}^2$? Justify your answer.

3 Define the *compactly supported cohomology* $H^*_{ct}(X;\mathbb{Z})$ of a space X, and compute $H^*_{ct}(\mathbb{R}^n;\mathbb{Z})$.

State a version of the Poincaré duality theorem for an oriented manifold M of dimension n. By considering the complement of a ball in M, or otherwise, show that if M is closed and oriented there is a map $f : M \to \mathbb{S}^n$ of degree 1. Is there always a map $f : \mathbb{S}^n \to M$ of degree 1? Justify your answer.

Finally, giving careful statements of any other standard results you use, show that the fibre bundle $\mathbb{S}^2 \to \mathbb{CP}^3 \to \mathbb{S}^4$ (which you may assume exists) does not admit a section.

4 Give a careful statement of the Thom isomorphism theorem for oriented vector bundles. Explain how to associate to a closed co-oriented submanifold Y of a smooth manifold M a cohomology class $\varepsilon_Y \in H^*_{ct}(M;\mathbb{Z})$.

Suppose now M is closed. Using a formula for $\varepsilon_\Delta \in H^*(M \times M;\mathbb{Q})$ without proof, or otherwise, show that a map $f : M \to M$ with

$$\sum_{k \geqslant 0} (-1)^k \mathrm{tr}(f^k : H^k(M;\mathbb{Q}) \to H^k(M;\mathbb{Q})) \neq 0$$

must have a fixed point. Finally, for every $g \geqslant 0$ give (and justify) examples of

(i) a closed oriented surface S of genus g with involution $\iota : S \to S$ such that every map homotopic to ι has fixed points;

(ii) a closed oriented surface S of genus g with involution $\iota : S \to S$ such that ι has fixed points but is homotopic to a map without fixed points.

5 Define the homotopy groups $\pi_k(X)$ of a topological space X. What is an Eilenberg-Maclane space $K(G,n)$? Prove that $K(G,n) \times K(H,n) \simeq K(G \times H, n)$.

Stating any general results you use without proof, describe a space $K(\mathbb{Z}_p, 1)$ and compute $H^*(K(\mathbb{Z}_p,1);\mathbb{Z}_p)$. *Assuming* the Künneth theorem may be applied, give a formula for $H^*(K(\mathbb{Z}_p \times \mathbb{Z}_p, 1);\mathbb{Z}_p)$.

Let X be obtained from an n-dimensional cell complex M by adding a single cell of dimension $n+1$ and at most finitely many cells of each dimension $i \geqslant n+2$. Show that $H^{n+1}(X;\mathbb{Z}_p)$ is either 0 or $\mathbb{Z}_p$.

Now suppose the finite group G acts freely on $\mathbb{S}^n$, with $n > 1$, and let $M = \mathbb{S}^n/G$ denote the quotient space. By adding cells to kill the non-zero elements of homotopy groups $\pi_i(M)$, with $i \geqslant n$, show that G cannot be isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

[*For the last stage, you may assume that (i)* $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ *and (ii) all the homotopy groups of spheres are finitely generated.*]

END OF PAPER