UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 9 to 12

PAPER 26

ALGEBRAIC TOPOLOGY

Attempt **FIVE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Show that $\mathbb{C}P^3$ and $S^2 \times S^4$ have isomorphic homology groups with $\mathbb{Z}$ coefficients, but are not homotopy equivalent.

(b) Show that any continuous map $S^2 \times S^4 \to \mathbb{C}P^3$ has degree zero, meaning that the induced homomorphism

$$H_6(S^2 \times S^4, \mathbb{Z}) \to H_6(\mathbb{C}P^3, \mathbb{Z})$$

is zero.

2 Let X be a simply connected closed 4-manifold with Betti number b_2 equal to 1. Show that the integral cohomology ring of X is isomorphic to that of $\mathbb{C}P^2$.

3 Write out the long exact sequence which relates the homology groups of a space B, a subspace $A \subset B$, and the relative homology groups. Define the homomorphisms in the sequence; in particular, show that the boundary homomorphism is well-defined. Show that the sequence is exact at the homology groups $H_i(A)$. Be sure to prove the algebraic results which you use.

4 Show that if there is a continuous map of non-zero degree from a closed oriented surface X to a closed oriented surface Y, then the genus of X is at least that of Y.

5 (a) Show that any continuous vector field on an even-dimensional sphere is zero at some point.

(b) Show that the sphere of any odd dimension has a continuous vector field which is non-zero at every point.

6 Show that if two ellipses in $\mathbb{R}^2$ intersect transversely, then their intersection consists of an even number of points. Also, show that their intersection consists of at most 4 points.