UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 1.30 to 4.30

PAPER 13

ALGEBRAIC TOPOLOGY

*Attempt **FIVE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Show that every simply-connected closed 3-manifold has the same homology groups as the 3-sphere.

(b) Show that any map $S^3 \to S^1 \times S^2$ has degree zero, meaning that the induced homomorphism

$$H_3(S^3, \mathbb{Z}) \to H_3(S^1 \times S^2, \mathbb{Z})$$

is zero.

2 Let X denote a connected cell complex whose universal cover is contractible. Let R denote the group ring $\mathbb{Z}[\pi_1(X, x)]$ for some base point x. Show that

$$H^i(X, \mathbb{Z}) \cong Ext^i_R(\mathbb{Z}, \mathbb{Z})$$

for all $i \geqslant 0$.

3 Show that the Euler characteristic of a closed orientable 6-manifold is even. Is this true for nonorientable closed 6-manifolds?

Show that for every even number n, there is a closed orientable 6-manifold with Euler characteristic equal to n.

4 Write out the long exact sequence which relates the homology groups of a space X, a subspace Y, and the relative homology groups. Prove that the sequence is exact, using the definition of relative homology groups. Be sure to prove the algebraic results which you use.

5 (a) Show that any finite group which acts freely on an even-dimensional sphere has order at most 2.

(b) Show that the finite cyclic group $\mathbb{Z}/a$ can act freely on the sphere S^{2b-1} for any $a, b \geq 1$.

(c) Show that any finite group which acts freely on $\mathbb{R}^n$ must be trivial, for any $n \geq 1$.

6 Let A and B be the subsets of the real projective plane $\mathbb{R}P^2$ defined by $A = \{[x, y, z] : F(x, y, z) = 0\}$, and $B = \{[x, y, z] : g(x, y, z) = 0\}$, where F is a homogeneous real polynomial of degree a and g is a homogeneous real polynomial of degree b.

Show that, for generic polynomials f and g, $A \cap B$ is a finite set with

$$|A \cap B| \equiv ab \ (mod\ 2).$$

Show, for some choice of a and b, that this statement cannot be improved to $|A \cap B| = ab$.