UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2008 9.00 to 12.00

PAPER 21

ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

k is a fixed algebraically closed field.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Prove the theorem on the dimension of fibres, that is, if $\phi : X \to Y$ is a surjective regular map of quasi-projective algebraic varieties, then $\dim X_y \geq \dim X - \dim Y$ for any $y \in Y$.

(ii) Prove that any regular map $\phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ is a constant map if $m > n$.

[You can use this: if X is affine and $f \in k[X]$ such that $\emptyset \neq V_X(f) \neq X$, then each component of $V_X(f)$ has dimension $\dim X - 1$]

2 (i) Let $X \subset \mathbb{A}^3_k$ be the union of the three coordinate axes and $Y = V(t_1 t_2 (t_1 - t_2)) \subset \mathbb{A}^2_k$. Prove that X and Y are not isomorphic.

(ii) Let $X = V(f) \subset \mathbb{A}^n_k$ where $\deg f = 3$. Show that if X contains two distinct singular points x, x', then it also contains the line joining x, x'.

3 (i) Let X be a normal quasi-projective algebraic variety of dimension d. Show that the set of singular points of X is contained in a closed subset of dimension $\leq d - 2$.

(ii) Prove that $V(t_1^2 + t_2^2 + \ldots + t_n^2) \subset \mathbb{A}^n_k$ is normal if $n > 2$.

4 (i) Let X be a quasi-projective algebraic variety which is smooth at $x \in X$. Show that local parameters at x generate the maximal ideal m_x in $\mathcal{O}_{X,x}$ and show that any $f \in \mathcal{O}_{X,x}$ has an associated formal power series.

(ii) Let X, Y be quasi-projective algebraic varieties, $x \in X$ and $y \in Y$ such that the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as k-algebras. Show that there are neighborhoods $x \in U$ and $y \in V$ and an isomorphism $\phi : U \to V$ such that $\phi(x) = y$.

5 (i) Using the Riemann-Roch theorem, prove that there is a group structure on the set of points of an elliptic curve.

(ii) Let X be a smooth projective curve of genus g. Prove that there is a finite regular map $\phi : X \to \mathbb{P}^1$ of degree $\leq g + 1$.

(iii) Let X be a smooth projective curve and $x_1, \ldots, x_m \in X$ distinct points. Show that there is a rational function on X which is regular on $U = X - \{x_1, \ldots, x_m\}$ but not regular at x_i for all $1 \leq i \leq m$.

END OF PAPER