UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June, 2006 1.30 to 4.30

PAPER 17

ALGEBRAIC GEOMETRY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define what is meant by a prevariety being *separated* and by a variety being *complete*. Show that any projective variety is necessarily both separated and complete.

2 Let $\phi : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism of varieties, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and let $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. Describe the constructions of

(i) the $\mathcal{O}_X$-module $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}$,

(ii) the $\mathcal{O}_X$-module $\phi_* \mathcal{F}$, and

(iii) the $\mathcal{O}_Y$-module $\phi^* \mathcal{H}$.

Given an affine variety V and a $k[V]$-module M, describe the construction of the associated quasi-coherent sheaf $\tilde{M}$ on V with $\tilde{M}(V) = M$ — you may omit the proof that the sheaf conditions (A) and (B) hold. Assuming the fact that any quasi-coherent sheaf on an affine variety is of this form, interpret the constructions (i), (ii), (iii) above (in terms of modules over the appropriate rings) when the sheaves are quasi-coherent and ϕ is a morphism of affine varieties.

If now $\phi : Y \to X$ is a morphism of affine varieties and M is a module over $k[X]$, prove that $\phi_* \phi^* \tilde{M} \cong \tilde{M} \otimes_{\mathcal{O}_X} \phi_* \mathcal{O}_Y$.

[*The construction of the sheafification of a presheaf, and its properties, may be assumed throughout in this question, as may standard results from commutative algebra.*]

3 Quoting the elementary results on flabby sheaves that you need, describe briefly the construction of sheaf cohomology on a topological space X, via a particular choice of flabby resolutions. Deduce from your construction that, for $\mathcal{F}$ any flabby sheaf, the higher cohomology $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Suppose now that X is a variety and $\mathcal{F}$ an $\mathcal{O}_X$-module; we define a *rational section* of $\mathcal{F}$ to be an equivalence class of pairs (U, s), where U is an open dense subset of X and $s \in \mathcal{F}(U)$, under the equivalence relation $\sim$ defined by $(U, s) \sim (V, t)$ if there exists an open dense subset W of X with $W \subset U \cap V$ and $s|_W = t|_W$. Show that the set $\mathrm{Rat}(\mathcal{F})$ of rational sections of $\mathcal{F}$ forms a module over the ring of rational functions $\mathrm{Rat}(X)$. From now on, we suppose that X is irreducible and $\mathcal{F}$ is locally free; show that, for any $P \in X$, there is an inclusion map of the stalk $\mathcal{F}_P$ into $\mathrm{Rat}(\mathcal{F})$.

Suppose further that X is an irreducible *curve*. We denote by $\mathcal{R}(\mathcal{F})$ the constant sheaf on X corresponding to $\mathrm{Rat}(\mathcal{F})$, and define a sheaf $\mathcal{P}(\mathcal{F})$ on X by

$$\Gamma(U, \mathcal{P}(\mathcal{F})) = \bigoplus_{P \in U} \mathrm{Rat}(\mathcal{F})/\mathcal{F}_P \ ,$$

with the obvious restriction maps. Justify the fact that that $\mathcal{P}(\mathcal{F})$ is a sheaf, and prove that there is a short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{R}(\mathcal{F}) \to \mathcal{P}(\mathcal{F}) \to 0.$$

Find an example where the natural map

$$\mathrm{Rat}(\mathcal{F}) \to \bigoplus_{P \in X} \mathrm{Rat}(\mathcal{F})/\mathcal{F}_P$$

is not surjective. What happens when X is affine? Justify your answers.

4 Describe the construction of the invertible sheaves $\mathcal{O}_{\mathbf{P}^n}(m)$ on $\mathbf{P}^n$ (where $m \in \mathbf{Z}$). Letting $\pi : \mathbf{A}^{n+1}\backslash\{0\} \to \mathbf{P}^n$ denote the standard map, and U denote an open subset of $\mathbf{P}^n$, show that the non-zero elements of $\Gamma(U, \mathcal{O}_{\mathbf{P}^n}(m))$ may be identified as quotients of coprime homogeneous polynomials in $X_0, X_1, \ldots, X_n$, say F/G, with $G \neq 0$ and $\deg F - \deg G = m$, such that F/G defines a regular function on $\pi^{-1}(U)$.

Consider now the sheaf of regular 1-forms $\Omega^1_{\mathbf{P}^n}$. Suppose that $f = P/Q$ is a rational function given as the quotient of homogeneous polynomials of the same degree, with $Q \neq 0$, which is regular on an open set U. Show that, for each $0 \leq i \leq n$, there is a well-defined element $\partial f/\partial X_i$ of $\Gamma(U, \mathcal{O}_{\mathbf{P}^n}(-1))$. Deduce the existence of a sequence of morphisms

$$0 \to \Omega^1_{\mathbf{P}^n} \to \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbf{P}^n}(-1) \to \mathcal{O}_{\mathbf{P}^n} \to 0,$$

where the second of the unknown maps is defined by the recipe (suitably interpreted)

$$(g_0, g_1, \ldots, g_n) \mapsto \sum_{i=0}^{n} X_i\, g_i.$$

By reducing down to affine pieces, show that the sequence is a short exact sequence.

Quoting appropriate results concerning the dimension of $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m))$, find the dimension of $H^i(\mathbf{P}^n, \Omega^1_{\mathbf{P}^n})$ for all $0 \leq i \leq n$.

END OF PAPER