UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Tuesday 3 June 2008 9.00 to 12.00

PAPER 51

ADVANCED QUANTUM FIELD THEORY

*Attempt **THREE** questions*

*There are **FOUR** questions in total*

The questions carry equal weight

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The complex amplitude $K(q_1, q_0; T)$ is defined by the path integral over paths $q(t)$,

$$K(q_1, q_0; T) = \int \mathrm{d}[q]\; e^{iS[q]} ,$$

where

$$S[q] = \int_0^T \mathrm{d}t \, \left(\frac{1}{2} m\, \dot{q}^2 - V(q)\right) , \qquad q(0) = q_0 \,, \quad q(T) = q_1 \,.$$

Show that an approximation obtained by first letting $q(t) = q_c(t) + f(t)$, where $q_c(t)$ is a classical path obeying the required boundary conditions, gives

$$K(q_1, q_0; T) \approx D(T)\; e^{iS[q_c]} ,$$

where $D(T)$ may be related to the functional determinant of a suitable operator. Explain why this approximation is exact if $V(q)$ is quadratic in q and $D(T)$ is then independent of q_0, q_1.

Show how these results are consistent with the free expression, when $V = 0$,

$$K_0(q_1, q_0; T) = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} e^{i\frac{1}{2}m(q_1 - q_0)^2/T} \,. \qquad (*)$$

Obtain the result for K for motion in a gravitational field when $V(q) = -mgq$.

For general V show how $K(q_1, q_0; -iT)$ can be expanded in terms of contributions involving the energy eigenvalues E_n and associated wave functions $\psi_n(q)$ for the corresponding quantum Hamiltonian.

Consider a free particle on a circle so that $q \sim q + 2\pi n$. Explain why $K(q_1, q_0; T)$ can be written as

$$K(q_1, q_0; T) = \sum_n K_0(q_1 + 2\pi n, q_0; T) ,$$

where K_0 is defined in $(*)$. Use this result to determine the energy eigenvalues and wave functions.

[The identity

$$\sum_n e^{-\frac{1}{2}(x+2\pi n)^2/y} = \left(\frac{y}{2\pi}\right)^{\frac{1}{2}} \sum_n e^{-\frac{1}{2}n^2 y + inx} ,$$

is important.]

2 Let $\mathcal{L}(\phi, \partial\phi)$ be the Lagrangian density for a multi-component scalar field $\phi = (\phi_1, \phi_2, \ldots)$. Suppose $\mathcal{L}$ is invariant under transformations $\delta\phi = \epsilon_a t_a \phi$ for arbitrary infinitesimal constants ϵ_a and $\{t_a\}$ is a set of antisymmetric matrices acting on ϕ. Show how this can be extended to any local $\epsilon_a(x)$ if $\partial_\mu\phi \to D_\mu\phi = \partial_\mu\phi + A_{\mu a} t_a \phi$ if $\delta A_{\mu a} = -\partial_\mu \epsilon_a - f_{bca} A_{\mu b} \epsilon_c$ assuming $[t_a, t_b] = f_{abc} t_c$.

Let $S[\phi, J, A] = \int d^d x \left(\mathcal{L}(\phi, D\phi) + J \cdot \phi \right)$ for arbitrary $J(x)$ and $J \cdot \phi = J_i \phi_i$. Show that

$$\left((t_a\phi) \cdot \frac{\delta}{\delta\phi} + (t_a J) \cdot \frac{\delta}{\delta J} + \partial_\mu \frac{\delta}{\delta A_{\mu a}} + f_{abc} A_{\mu b} \frac{\delta}{\delta A_{\mu c}} \right) S[\phi, J, A] = 0 \,. \qquad (*)$$

For the corresponding quantum field theory we define

$$Z[J, A] = \int \mathrm{d}[\phi] \, e^{iS[\phi, J, A]} \,,$$

and then

$$Z[J, A] = e^{iW[J,A]} \,, \qquad \frac{\delta}{\delta J_i(x)} W[J, A] = \varphi_i(x) \,, \quad \Gamma[\varphi, A] = -W[J, A] + \int \mathrm{d}^d x \, J \cdot \varphi \,.$$

Let $\hat{\tau}_{i_1 \ldots i_n}(p_1, \ldots, p_n)$ be defined by

$$\int \prod_{r=1}^{n} \mathrm{d}^d x_r \, e^{ip_r \cdot x_r} \, \frac{\delta}{\delta\varphi_{i_1}(x_1)} \cdots \frac{\delta}{\delta\varphi_{i_n}(x_n)} \, \Gamma[\varphi, 0] \Big|_{\varphi=0} = (2\pi)^d \delta^d \big(\textstyle\sum_r p_r \big) \, \hat{\tau}_{i_1 \ldots i_n}(p_1, \ldots, p_n) \,.$$

Describe briefly the contributions to this amplitude in terms of Feynman diagrams.

Starting from the identity $(*)$ for S obtain a corresponding identity for Z and hence derive

$$\left((t_a\varphi) \cdot \frac{\delta}{\delta\varphi} + \partial_\mu \frac{\delta}{\delta A_{\mu a}} + f_{abc} A_{\mu b} \frac{\delta}{\delta A_{\mu c}} \right) \Gamma[\varphi, A] = 0 \,. \qquad (**)$$

[It is necessary to show that $\delta W / \delta A|_J = -\delta\Gamma / \delta A|_\varphi$ and $(t_a J) \cdot \varphi = -J \cdot (t_a \varphi)$.]

Define

$$\int \prod_{r=1}^{3} \mathrm{d}^d x_r \, e^{ip_r \cdot x_r} \, \frac{\delta}{\delta A_{\mu a}(x_1)} \frac{\delta}{\delta\varphi_i(x_2)} \frac{\delta}{\delta\varphi_j(x_3)} \, \Gamma[\varphi, A] \Big|_{\varphi, A=0} = (2\pi)^d \delta^d \big(\textstyle\sum_r p_r \big) \, \hat{\tau}^\mu_{a,ij}(p_1, p_2, p_3) \,.$$

Show from the identity $(**)$

$$p_{1\mu} \hat{\tau}^\mu_{a,ij}(p_1, p_2, p_3) = i(t_a)_{ik} \, \hat{\tau}_{kj}(p_1 + p_2, p_3) - \hat{\tau}_{ik}(p_2, p_3 + p_1) \, i(t_a)_{kj} \,. \qquad (\dagger)$$

Suppose $\mathcal{L}(\phi, D\phi) = -\frac{1}{2}\big((D^\mu\phi) \cdot (D_\mu\phi) + m^2 \phi \cdot \phi \big)$. Explain why

$$\hat{\tau}^\mu_{a,ij}(p_1, p_2, p_3) = -i(p_2 - p_3)^\mu \, (t_a)_{ij} \,,$$

is sufficient to verify $(\dagger)$ in this case.

3 Consider a quantum field theory with a single scalar field ϕ and a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - V(\phi)\,.$$

What does it mean to say that the theory is renormalisable? In four dimensions obtain restrictions on $V(\phi)$ which ensure that the theory is renormalisable. How is the bare Lagrangian density defined?

Suppose the theory has a single dimensionless coupling g and no mass parameters. Let $\langle\phi(x_1)\ldots\phi(x_n)\rangle$ be the finite correlation function determined by perturbation expansion of the quantum field theory as a series in g. Why must this also depend on an additional scale μ? Describe the derivation of the equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma(g)\right)\langle\phi(x_1)\ldots\phi(x_n)\rangle = 0\,,$$

and briefly discuss its interpretation.

Assuming

$$\int \mathrm{d}^4x\, e^{ip\cdot x}\langle\phi(x)\phi(0)\rangle = -i\,\frac{d(p^2/\mu^2, g)}{p^2}\,,$$

show how the behaviour of $d(p^2/\mu^2, g)$ for large p^2 depends on the form of $\beta(g)$. If $\beta(g) = -bg^3,\ \gamma(g) = cg^2$ with $b > 0$ find an expression for $d(p^2/\mu^2, g)$ for large p^2. If $\beta(g) = -bg^3 - ag^5$ and $b < 0,\ a > 0$ what happens for large p^2?

4 Explain why gauge fixing is necessary for obtaining a perturbative expansion in quantum gauge theories. Suppose for a gauge theory the quantum action is

$$S_q[A,c,\bar{c}] = -\frac{1}{g^2}\int \mathrm{d}^d x \left(\frac{1}{4}\, F^{\mu\nu}\cdot F_{\mu\nu} + \frac{1}{2\xi}\,\partial^\mu A_\mu \cdot \partial^\nu A_\nu + \partial^\mu \bar{c}\cdot D_\mu c \right),$$

where $A_{\mu a}$ is a gauge field, $c_a, \bar{c}_a$ are ghost fields and

$$F_{\mu\nu\, a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc} A_{\mu b} A_{\nu c}\,, \quad (D_\mu c)_a = \partial_\mu c_a + f_{abc} A_{\mu b} c_c\,.$$

For a perturbation expansion in g derive expressions for the the free field propagators defined by

$$\int \mathrm{d}^d x\; e^{-ip\cdot x} \langle A_{\mu a}(x)\, A_{\nu b}(0)\rangle = \delta_{ab}\, i\tilde{\Delta}_{F\mu\nu}(p)\,,$$
$$\int \mathrm{d}^d x\; e^{-ip\cdot x} \langle c_a(x)\, \bar{c}_b(0)\rangle = \delta_{ab}\, i\tilde{\Delta}_{F}(p)\,.$$

Verify that $\tilde{\Delta}_{F\mu\nu}(p)p^\nu = \xi\, p_\mu \tilde{\Delta}_F(p)$.

By isolating the relevant term in S_q show that the Feynman rules for a vertex involving the fields $\bar{c}_a A_{\mu b} c_c$ require a contribution $g\, p_\mu f_{abc}$, where p_μ is the incoming momentum on the $\bar{c}$ line.

Assuming $\xi = 1$ write down an expression for the one loop contribution to $\int \mathrm{d}^d x\; e^{-ip\cdot x}\langle c_a(x)\, \bar{c}_b(0)\rangle$. Show that it involves the integral

$$\frac{1}{(2\pi)^d i}\int \mathrm{d}^d k\; \frac{p\cdot k}{((p-k)^2 - i\epsilon)(k^2 - i\epsilon)} = \frac{(p^2)^{\frac{1}{2}d-1}}{(4\pi)^{\frac{1}{2}d}}\, \Gamma(2-\tfrac{1}{2}d) \int_0^1 \mathrm{d}\alpha\, \alpha^{\frac{1}{2}d-2}(1-\alpha)^{\frac{1}{2}d-1}\,.$$

Sketch how this result for the Feynman integral is obtained and using dimensional regularisation determine the divergent part of this amplitude. How is this divergence cancelled by introducing a counterterm in the quantum action?

[You may use $f_{acd} f_{bcd} = C\,\delta_{ab}$, with C a group theory constant. $\Gamma(a)$ is here the standard Gamma function, $\Gamma(1) = 1$, $\Gamma(a+1) = a\Gamma(a)$.]

END OF PAPER