UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 6 June, 2006 9 to 12

# PAPER 49

# ADVANCED QUANTUM FIELD THEORY

*Attempt* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Assuming

$$\langle q'|e^{-i\frac{1}{2}\hat{p}^2 T}|q\rangle = \left(\frac{1}{2\pi i T}\right)^{\frac{1}{2}} e^{i\frac{(q'-q)^2}{2T}} ,$$

where $\hat{q}, \hat{p}$ are the usual quantum mechanical position, momentum operators with $\hat{q}|q\rangle = q|q\rangle$, obtain the path integral representation involving a functional integral over paths $q(t)$ with $q(0) = q\,,\ q(T) = q'$,

$$\langle q'|e^{-i\hat{H}T}|q\rangle = \int \mathrm{d}[q]\, e^{iS[q]} ,$$

where the Hamiltonian $\hat{H}$ and the action $S[q]$ are given by

$$\hat{H} = \tfrac{1}{2}\,\hat{p}^2 + V(\hat{q})\,, \qquad S[q] = \int_0^T \mathrm{d}t\,\big(\tfrac{1}{2}\,\dot{q}^2 - V(q)\big)\,.$$

For $V(q)$ quadratic in $q$, show that

$$\langle q'|e^{-i\hat{H}T}|q\rangle = N\, e^{iS[q_c]} ,$$

where $q_c$ is the classical path linking $q$ and $q'$ and $N$ is independent of $q, q'$. Show that $N$ can be expressed in terms of the determinant of the operator $\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''(q_c)$ acting on functions $f(t)$ with $f(0) = f(T) = 0$.

Consider the action

$$S[q] = \int_{-\infty}^{\infty} \mathrm{d}t\,\big(\tfrac{1}{2}\,\dot{q}^2 - \tfrac{1}{2} m^2 q^2 + Jq\big)\,,$$

for arbitrary $J(t)$. Show that the functional integral over appropriate paths $q(t)$ with $-\infty < t < \infty$ can in this case be expressed the form

$$\int \mathrm{d}[q]\, e^{iS[q]} = N\, e^{-\frac{1}{2}i\int \mathrm{d}t\mathrm{d}t'\, J(t)G(t-t')J(t')}\,, \quad G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{i\omega t}\,\frac{1}{\omega^2 - m^2 + i\epsilon}\,.$$

**2** For a scalar field theory in $d$-dimensions with Lagrangian density

$$\mathcal{L}(\phi) = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\,\lambda\phi^4\,,$$

define the generating functional $Z[J]$ and also the corresponding functionals $W[J]$ and $\Gamma[\varphi]$ where $\delta W/\delta J = \varphi$. Assume that for $J=0$ then $\varphi = 0$.

Let

$$\int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, e^{ip_1\cdot x_1 + \ldots + p_n \cdot x_n} \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)}\bigg|_{\varphi=0}$$
$$= (2\pi)^d \delta^d(p_1 + \ldots + p_n)\,\tau_n(p_1, \ldots, p_n)\,.$$

Show that, for $\lambda = 0$, $\tau_2(p,-p) = p^2 + m^2$ and $\tau_n = 0$ for $n \geq 3$. If $\lambda > 0$, show that for contributions corresponding to Feynman diagrams with zero loops $\tau_2$ is unchanged but $\tau_4(p_1,p_2,p_3,p_4) = \lambda$ and we may take $\Gamma[\varphi] = -\int \mathrm{d}^d x\, \mathcal{L}(\varphi)$.

Draw the Feynman diagrams representing the one loop contributions to $\tau_2$ and $\tau_4$. Evaluate the divergent part of the one loop contribution to $\tau_2$ for $d \approx 4$ and show that it can be cancelled by an additional counterterm $\mathcal{L}_{\text{c.t.}} = -\frac{1}{2}B\phi^2$ for suitable $B$.

[You may assume $\int \mathrm{d}^d k\, e^{-\alpha k^2} = (\pi/\alpha)^{\frac{1}{2}d}$ for $\alpha > 0$ and $k^2 = \sum_{i=1}^d k_i^2$, and also $\int_0^\infty \mathrm{d}s\, s^{a-1} e^{-s} = \Gamma(a)$ defines the standard Gamma function.]

**3** Consider a quantum field theory with a single scalar field and a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{1}{n!}\,g_n\phi^n\,.$$

In $d$ spacetime dimensions what is the dimension of $\phi$? Show that the coupling $g_n$ has dimension $d_n = n - d(\frac{1}{2}n - 1)$. For a Feynman diagram with $L$ loops, $I$ internal lines, $E$ external lines and $V_n$ vertices with $n$ lines incident show that the overall degree of divergence of the Feynman integral can be expressed as

$$D = d - (\tfrac{1}{2}d - 1)E - \sum_n d_n V_n\,.$$

Comment on the distinction between renormalisable and nonrenormalisable theories for $d = 3, 4$. Why does considering $d$ to be an arbitrary complex parameter provide a prescription for regularising the theory?

Suppose $g_n = 0$ for $n \neq 4$ and $g_4 = \mu^\varepsilon\lambda$ with $\lambda$ a dimensionless coupling, $\varepsilon = 4 - d$, $\mu$ an arbitrary mass scale. Describe in outline how the functional integral with dimensional regularisation leads to a finite perturbation expansion for the four dimensional quantum field theory in terms of the action

$$S_0 = \int \mathrm{d}^d x\,\mathcal{L}_0\,, \qquad \mathcal{L}_0 = -\tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - \tfrac{1}{24}\lambda_0\phi_0^4\,,$$

where $\phi_0 = Z^{\frac{1}{2}}\phi$, $m_0^2 = Z_{m^2}m^2$, $\lambda_0 = \mu^\varepsilon Z_\lambda\lambda$. Here $Z, Z_{m^2}, Z_\lambda$ depend on $\lambda, \varepsilon$ with $Z, Z_{m^2}, Z_\lambda = 1 + \mathrm{O}(\lambda)$. It is sufficient to indicate why $F(\lambda, m^2, h; \mu)$ defined by

$$e^{iF(\lambda,m^2,h;\mu)} = \frac{1}{N_0}\int \mathrm{d}[\phi]\; e^{iS_0[\phi] + \mu^{-\frac{1}{2}\varepsilon}h\int \mathrm{d}^d x\,\phi}\,,$$

where $N_0$ is chosen so that $F = 0$ for $h = 0$, is a finite function of $\lambda, m^2, h$. Explain why

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}F = 0 \quad \text{for constant} \quad \lambda_0, m_0^2, h_0 \quad \text{where} \quad h_0 = \mu^{-\frac{1}{2}\varepsilon}Z^{-\frac{1}{2}}h\,.$$

Hence obtain the RG equation

$$\left(\mu\frac{\partial}{\partial\mu} + \hat\beta_\lambda\frac{\partial}{\partial\lambda} + \gamma_{m^2}m^2\frac{\partial}{\partial m^2} + \hat\gamma_h h\frac{\partial}{\partial h}\right)F(\lambda, m^2, h; \mu) = 0\,,$$

for suitable $\hat\beta_\lambda, \gamma_{m^2}, \hat\gamma_h$ depending on $\lambda$.

Why may we write $F(\lambda, m^2, h; \mu) = \hat F(\lambda, m^2/\mu^2, h/\mu^3)$? Show how the RG equation may be solved for $m^2 = 0$ in terms of a suitable $\lambda(\mu), h(\mu)$. Explain what is meant by a UV fixed point.

**4** Explain why gauge fixing is necessary in order to obtain a perturbative expansion for the quantum field theory for a non abelian gauge theory. Suppose the Lagrangian density after gauge fixing has the form

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}{}_a F_{\mu\nu a} + b_a \partial^\mu A_{\mu a} + \tfrac{1}{2}\xi\, b_a b_a + \bar{c}_a \partial^\mu (D_\mu c)_a \,,$$
$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc} A_{\mu b} A_{\nu c} \,, \quad (D_\mu c)_a = \partial_\mu c_a + f_{abc} A_{\mu b} c_c \,,$$

where $A_{\mu a}$ is the gauge field with group index $a$, $\bar{c}_a, c_a$ are ghost fields, $b_a$ is an auxiliary scalar field and $f_{abc}$ are the antisymmetric structure constants for the associated Lie algebra. Show that $s\mathcal{L} = 0$ and $s^2 = 0$ where the action of $s$ is defined by

$$sA_{\mu a} = (D_\mu c)_a \,, \qquad sc_a = -\tfrac{1}{2} f_{abc} c_b c_c \,, \qquad s\bar{c}_a = -b_a \,, \qquad sb_a = 0 \,,$$

and also $s(c_a X) = (sc_a)X - c_a\, sX$ and similarly for $c \to \bar{c}$.

Assume $Q_{BRS}$, satisfying ${Q_{BRS}}^2 = 0$, ${Q_{BRS}}^\dagger = Q_{BRS}$, is the corresponding operator acting on states in the quantum field theory. Explain how a space of physical states is constructed. Suppose $|A_{\mu a}(k)\rangle$, $|c_a(k)\rangle$, $|\bar{c}_a(k)\rangle$ are single particle states with 4-momentum $k$ and with non zero scalar products

$$\langle A_{\mu a}(k) | A_{\nu b}(k') \rangle = \eta_{\mu\nu} \delta_{ab}\, \delta_{kk'} \,, \qquad \langle \bar{c}_a(k) | c_b(k') \rangle = \delta_{ab}\, \delta_{kk'} \,.$$

If

$$Q_{BRS} |A_{\nu a}(k)\rangle = \alpha k_\nu |c_a(k)\rangle \,, \qquad Q_{BRS} |\bar{c}_a(k)\rangle = \bar{\alpha} k^\mu |A_{\mu a}(k)\rangle \,,$$

what are the conditions following from BRS symmetry. Determine the physical single particle states and show that they have positive norm.

## END OF PAPER