UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 9 to 12

PAPER 62

ADVANCED QUANTUM FIELD THEORY

*Attempt **THREE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A non-relativistic particle in one dimension has position $q(t)$, momentum $p(t)$ and Hamiltonian $H = \frac{p^2}{2m} + V(q)$. State how the quantum amplitude for finding $q(T) = \beta$ given $q(0) = \alpha$ can be written as a functional integral over paths in (i) phase space (with coordinates q, p); and (ii) position space. Show, by using precise definitions for the functional integrals, that the expressions in (i) and (ii) are equivalent and that they coincide with the result for the amplitude obtained using canonical, or operator, quantization.

2 A scalar field theory has classical action $S[\phi]$. Explain how quantum correlation functions can be obtained from a generating functional $Z[J]$. Give a formula for $W[J]$, the generating functional for connected correlation functions, and check this by obtaining an expression for the full four-point function in terms of connected correlation functions, assuming $\delta W/\delta J(x) = 0$ when $J(x) = 0$.

Define the quantum effective action $\Gamma[\phi]$ and derive a relation between $\delta^2 W/\delta J(x)\delta J(y)$ and $\delta^2\Gamma/\delta\phi(x)\delta\phi(y)$.

Calculate $W[J]$ for a free field of mass m, expressing your result in terms of the Feynman free propagator Δ_F and commenting on the origin of the $i\epsilon$ prescription in this approach. Hence show that $\Gamma[\phi] = S[\phi]$ for this theory.

3 A scalar field ϕ in d spacetime dimensions has mass m and interaction lagrangian $\mathcal{L}_I(\phi) = -\lambda\phi^n$. Explain what is meant by the superficial degree of divergence Δ of any Feynman diagram arising in perturbation theory and derive a formula for Δ involving the number of external lines of the diagram, the order in λ at which it contributes, and the mass dimension of λ.

Describe very briefly what is meant by dimensional regularization and minimal subtraction. Illustrate your answers by calculating the divergent part of the one-loop, order λ^2 contibutions to the two-point function or propagator in a scalar field theory, as above, with $d = 6$ and $n = 3$.

[You may use $\Gamma(z-1) = -\frac{1}{z} + (\gamma_E - 1) + O(z)$.]

4 Define gauge transformations acting on vector fields A_μ^a associated with some Lie algebra and describe how to construct the gauge-invariant Yang-Mills lagrangian $\mathcal{L}(A_\mu^a)$. [You should derive the gauge transformation properties of any additional quantities that may be relevant and explain any notation you introduce.]

Define the BRST operation s on the fields A_μ^a and also on the new fields c^a, $\bar{c}^a$ and b^a whose natures you should specify. State, without proof, two properties of s which imply that

$$\mathcal{L}(A_\mu^a) + s\Psi(A_\mu^a, c^a, \bar{c}^a, b^a) \qquad (*)$$

is BRST invariant for any Ψ. Show that for a suitable, simple choice of Ψ the integration over the field b^a in $(*)$ can be carried out to produce a new lagrangian

$$\mathcal{L}(A_\mu^a) - \tfrac{1}{2}(\partial^\mu A_\mu^a)^2 - \bar{c}^a \partial_\mu D^\mu c^a \qquad (**)$$

for the remaining fields. Explain the difficulty which arises in attempting to obtain Feynman rules from $\mathcal{L}(A_\mu^a)$ alone and how this is resolved for the lagrangian $(**)$.