UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 9 to 12

## PAPER 32

## ADVANCED PROBABILITY

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** In this exercise, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n, n \geqslant 0), P)$, and all definitions are understood with respect to this filtered space.

a) Let $(X_n, n \geqslant 0)$ be a submartingale which is bounded in $L^1$.

(i) Prove that for every $n \geqslant 0$, the sequence $(E[X_p^+ | \mathcal{F}_n], p \geqslant n)$ is increasing and converges to an a.s. limit $M_n$.

(ii) Show that $(M_n, n \geqslant 0)$ is a non-negative martingale which is bounded in $L^1$, and conclude that $X_n$ can be written in the form $M_n - Y_n$, where $(Y_n, n \geqslant 0)$ is a non-negative supermartingale which is bounded in $L^1$.

b) Let $(X_n, n \geqslant 0)$ be a supermartingale which is bounded in $L^1$. Show that $X_n$ can be written in the form $M_n + Y_n$, where $(M_n, n \geqslant 0)$ is a uniformly integrable martingale, and $(Y_n, n \geqslant 0)$ is a supermartingale with limit 0 when $n \to \infty$.

**2** Let $\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \mathbb{R}, i \geqslant 1\}$ be the set of real-valued sequences. For $\omega \in \Omega$ and $n \geqslant 1$ we let $X_n(\omega) = \omega_n$, and we let $S_0 = 0$, $S_n = X_1 + \ldots + X_n$. We define $\mathcal{F}_n = \sigma(X_k, 1 \leqslant k \leqslant n)$ and $\mathcal{F} = \mathcal{F}_\infty$.

Let $\mu$ be a probability measure on $\mathbb{R}$. We let $P$ be the unique measure on $(\Omega, \mathcal{F})$ under which the sequence $X_1, X_2, \ldots$ is independent and identically distributed with common distribution $\mu$. We let $P_n$ be the restriction of $P$ to $\mathcal{F}_n$.

For $\lambda \geqslant 0$ we let $\phi(\lambda) = E[e^{\lambda X_1}]$, where $E$ is the expectation associated with $P$. We assume that $\phi(\lambda)$ is finite for every $\lambda \geqslant 0$.

a) Show that under $P$, for every $\lambda \geqslant 0$ the process $M^\lambda = (\exp(\lambda S_n)/\phi(\lambda)^n, n \geqslant 0)$ is an $(\mathcal{F}_n, n \geqslant 0)$-martingale.

b) Let $P_n^\lambda$ be the probability measure on $(\Omega, \mathcal{F}_n)$ which is absolutely continuous with respect to $P_n$ with density

$$\frac{\mathrm{d}P_n^\lambda}{\mathrm{d}P_n} = M_n^\lambda.$$

Show that under $P_n^\lambda$, the random variables $X_1, \ldots, X_n$ are independent and identically distributed. Identify their common distribution $\mu^\lambda$, and show that it has mean $\phi'(\lambda)/\phi(\lambda)$.

c) In this part, we assume that $\mu$ is supported by $\mathbb{Z}_- \cup \{1\} = \{\ldots, -3, -2, -1, 0, 1\}$. For $k \geqslant 0$ let

$$\tau_k = \inf\{n \geqslant 0 : S_n \geqslant k\}.$$

We let $P^\lambda$ be the unique probability distribution on $(\Omega, \mathcal{F})$ under which $(X_n, n \geqslant 1)$ is independent and identically distributed with common distribution $\mu^\lambda$, and we let $E^\lambda$ be the expectation associated with $P^\lambda$.

Show that

$$P(\tau_k \leqslant n) = E^\lambda[(M_n^\lambda)^{-1} \mathbb{1}_{\{\tau_k \leqslant n\}}] = E^\lambda[(M_{\tau_k}^\lambda)^{-1} \mathbb{1}_{\{\tau_k \leqslant n\}}] = e^{-\lambda k} E^\lambda[\phi(\lambda)^{\tau_k} \mathbb{1}_{\{\tau_k \leqslant n\}}].$$

Assuming that there exists $\lambda_0 > 0$ such that $\phi(\lambda_0) = 1, \phi'(\lambda_0) > 0$, compute $P(\tau_k < \infty)$, and deduce the law of $\sup_{n \geqslant 0} S_n$ under $P$.

**3** Let $(M_t, t \geqslant 0)$ be a continuous-time martingale with respect to a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geqslant 0), P)$, such that $(M_t, t \geqslant 0)$ is a non-negative process with continuous paths, and which converges a.s. to 0 as $t \to \infty$. Let $M^* = \sup_{t \geqslant 0} M_t$. We use the notation $P(A|\mathcal{G}) = E[\mathbb{1}_A|\mathcal{G}]$, where $A$ is an event and $\mathcal{G}$ a sub-$\sigma$-algebra of $\mathcal{F}$.

a) Show that for every $x > 0$,

$$P(M^* \geqslant x|\mathcal{F}_0) = 1 \wedge (M_0/x).$$

[Hint: Use the stopped martingale $(M_{t \wedge T_x}, t \geqslant 0)$, where $T_x = \inf\{t \geqslant 0 : M_t \geqslant x\}$.]

b) Deduce that $M^*$ has the same distribution as $M_0/U$, where $U$ is uniform on $[0, 1]$ and independent of $M_0$.

c) Let $(B_t, t \geqslant 0)$ be a Brownian motion started at $B_0 = a > 0$. Give the distribution of $\sup_{0 \leqslant t \leqslant T_0} B_t$, where $T_0 = \inf\{t \geqslant 0 : B_t = 0\}$.

**4** State and prove the reflection principle for the standard 1-dimensional Brownian motion.

Let $(B_t, t \geqslant 0)$ be a standard 1-dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$. Use the reflection principle to show that $S_t = \sup_{0 \leqslant s \leqslant t} B_s$ has the same law as $|B_t|$ for every $t \geqslant 0$.

Let $a < b < c < d$ be non-negative real numbers. Show that

$$P\left(\sup_{a \leqslant t \leqslant b} B_t = \sup_{c \leqslant t \leqslant d} B_t\right) = 0.$$

**5** Let $(B_t, t \geqslant 0)$ be a standard 1-dimensional Brownian motion. For $x \in \mathbb{R}$, let $T_x = \inf\{t \geqslant 0 : B_t = x\}$. Fix $a, b > 0$, and let $T = T_a \wedge T_{-b}$.

By considering processes of the form $(\exp(\lambda B_t - \lambda^2 t/2), t \geqslant 0)$, or otherwise, prove that for every $\lambda \in \mathbb{R}$,

$$E(e^{-\lambda^2 T/2} \mathbb{1}_{\{T = T_a\}}) = \frac{\sinh(\lambda b)}{\sinh(\lambda(a + b))},$$

and that

$$E(e^{-\lambda^2 T/2}) = \frac{\cosh(\lambda(a - b)/2)}{\cosh(\lambda(a + b)/2)}.$$

**6** a) Write carefully the definition of a Poisson random measure on a measurable space $(E, \mathcal{E})$, with $\sigma$-finite intensity $\mu(\mathrm{d}x)$.

b) Fix $d \geqslant 1$, and let $\lambda(\mathrm{d}x)$ be Lebesgue measure on $\mathbb{R}^d$. We let $B(0, r)$ be the open Euclidean ball in $\mathbb{R}^d$ with centre 0 and radius $r \geqslant 0$, and we let $v_d = \lambda(B(0, 1))$.

Let $M(\mathrm{d}x)$ be a Poisson random measure on $\mathbb{R}^d$ with intensity $\lambda(\mathrm{d}x)$. If $f$ is a non-negative measurable function and $\nu$ is a non-negative measure, we let $\nu(f) = \int f \mathrm{d}\nu$.

(i) Let $R = \sup\{r \geqslant 0 : M(B(0, r)) = 0\}$. Show that the distribution of $R$ has a density and compute it.

(ii) Let $N_r = M(B(0, r))$ for $r \geqslant 0$. Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be a continuous function with compact support. Compute $E[N_r \exp(-M(f))]$.

(iii) Show that the two quantities

$$E[\exp(-M(f))|N_r \geqslant 1] \qquad \text{and} \qquad \frac{E[N_r \exp(-M(f))]}{P(N_r \geqslant 1)}$$

have the same limit as $r \downarrow 0$, and compute this limit.

## END OF PAPER