UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Monday 6 June, 2005 9 to 12

PAPER 34

ADVANCED PROBABILITY

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $M = (M_n)_{n\geq 0}$ be a discrete-time random process, which is integrable, and adapted to a filtration $(\mathcal{F}_n)_{n\geq 0}$. Show that the following are equivalent:

(i) M is a martingale,

(ii) $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ for all bounded stopping times T.

(b) Assume that M is a martingale and that $M_n \to M_\infty$ a.s. as $n \to \infty$. State an additional condition, expressible in terms of the laws $\mu_n(dx) = \mathbb{P}(M_n \in dx)$, which would allow us to conclude that $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ for all, possibly infinite, stopping times T.

(c) Let $(Z_n)_{n\geq 1}$ be a sequence of independent $\mathrm{N}(0,1)$ random variables and let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Set $M_0 = 0$ and define

$$M_n = \sum_{k=1}^{n} a_k Z_k, \quad n \geq 1.$$

By consideration of characteristic functions, or otherwise, show that M_n converges a.s. only if $\sum_{k=1}^{\infty} a_k^2 < \infty$.

(d) Under what additional conditions if any on the sequence $(a_n)_{n\geq 1}$ can we conclude that $\mathbb{E}(M_T) = 0$ for $T = \inf\{n \geq 0 : M_n \geq 1\}$?

2 (a) State the almost-sure martingale convergence theorem.

(b) Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function and define for $n \in \mathbb{N}, k \in \{0, 1, \ldots, 2^n - 1\}$ and $\omega \in [k2^{-n}, (k+1)2^{-n})$,

$$X_n(\omega) = 2^n\{f((k+1)2^{-n}) - f(k2^{-n})\}.$$

Show that, for a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a suitable filtration $(\mathcal{F}_n)_{n\geq 0}$, the sequence $(X_n)_{n\in\mathbb{N}}$ may be considered as a martingale.

(c) Deduce that there exists a bounded measurable function $\dot{f} : [0,1] \to \mathbb{R}$ such that, for all $a, b \in [0,1]$ with $a \leq b$, we have

$$\int_a^b \dot{f}(x)dx = f(b) - f(a).$$

3 (a) Let $X = (X_t)_{t \in I}$ be a random process indexed by the set I of dyadic rationals in the interval $[0,1]$. Let $p \geq 1$ and $\beta > 1/p$ and suppose that

$$\|X_s - X_t\|_p \leq C|s-t|^{\beta}, \quad \text{for all } s,t \in I,$$

for some constant $C < \infty$. Show that, for any $\alpha \in [0, \beta - (1/p))$, setting

$$K_\alpha = 2 \sum_{n=0}^{\infty} 2^{n\alpha} \sup_{k=0,1,\ldots,2^n-1} |X_{(k+1)2^{-n}} - X_{k2^{-n}}|,$$

we have

(i) $|X_s - X_t| \leq K_\alpha |s-t|^\alpha$ for all $s,t \in I$,

(ii) $K_\alpha \in L^p(\mathbb{P})$.

(b) Explain the rôle which this fact can play in the construction of Brownian motion and in determining the regularity of the sample paths of Brownian motion.

4 (a) Let $B = (B_t)_{t \geq 0}$ be a Brownian motion in $\mathbb{R}^d, d \geq 3$, starting from x. Fix $\varepsilon > 0$ and set

$$T = \inf\{t \geq 0 : |B_t| \leq \varepsilon\}.$$

Assume that $|x| > \varepsilon$. Show that

$$\mathbb{P}_x(T < \infty) = (\varepsilon/|x|)^{d-2}.$$

(b) For $t > 0$ and $x, y \in \mathbb{R}^d$, set

$$p(t, x, y) = (2\pi t)^{-d/2} e^{-|x-y|^2/2t}.$$

By evaluating the integral

$$I = \int_0^\infty \int_{\mathbb{R}^d} p(t, x, y) p(s, 0, y) dy ds$$

in two different ways, establish the identity

$$\int_{\mathbb{R}^d} p(t, x, y) |y|^{2-d} dy = c_d \int_t^\infty p(s, 0, x) ds,$$

where c_d is given by

$$c_d = \int_0^\infty (2\pi s)^{-d/2} e^{-1/2s} ds.$$

(c) Show that, for $x \neq 0$, as $\varepsilon \to 0$, we have

$$\varepsilon^{2-d} \mathbb{P}_x(T \leq t) \to c_d \int_0^t p(s, 0, x) ds.$$

5 (a) Let W be a Brownian motion in $\mathbb{R}^n, n \geq 1$, starting from 0, and let U be a random variable in $\mathbb{R}^n$ which is uniformly distributed on the unit ball $\{|x| \leq 1\}$ and is independent of W. Set $T = \inf\{t \geq 0 : |W_t| = |U|\}$. Show that W_T has the same distribution as U.

(b) Suppose now that W starts from a general point x in some connected open set D in $\mathbb{R}^n$. Set

$$g_D(x) = \mathbb{E}_x(T_D), \quad x \in D,$$

where $T_D = \inf\{t \geq 0 : W_t \notin D\}$. Show that if $g_D(x) < \infty$ for some $x \in D$ then $g_D(y) < \infty$ for all $y \in D$.

(c) For $n = 1, 2, 3$ and for $D = D_n = (0, \infty)^n$, determine whether g_D is finite.

6 (a) Let μ be a Poisson random measure on $\mathbb{R} \times (0, \infty)$ with intensity

$$\nu(dy, dt) = K(dy)dt = c|y|^{-2}dydt,$$

where $c \in (0, \infty)$ is determined by

$$2c \int_0^\infty \frac{(1 - \cos z)}{z^2} dz = 1.$$

Set

$$X_t = \int_{(0,t] \times \{|y| \leq 1\}} y(\mu - \nu)(dy, ds) + \int_{(0,t] \times \{|y| > 1\}} y\mu(dy, ds).$$

Explain why these integrals are well-defined in spite of the fact that

$$\int_{\{|y| \leq 1\}} yK(dy) = \int_{\{|y| > 1\}} yK(dy) = \infty.$$

(b) Write down the characteristic function of X_1 and hence obtain the density function of X_1.

(c) Fix $\alpha \in (0, \infty)$ and set $X_t^{(\alpha)} = \alpha X_{\alpha t}$. Show that the processes $X^{(\alpha)}$ and X have the same distribution.

END OF PAPER