UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 12

PAPER 27

ADVANCED PROBABILITY

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a Lévy process with characteristic exponent ψ. Let X_t be such a process. Show that, for all $u \in \mathbb{R}$, the following process is a martingale:

$$M_t^u = \exp\{iuX_t - t\psi(u)\}.$$

What is an infinitely divisible distribution? Let X_t be a Lévy process. Show that the distribution of X_1 is infinitely divisible.

State carefully, without proof, the Lévy-Khinchin theorem.

Let X_t be a continuous Lévy process. Show that it is expressible in the form $bt + \sqrt{a}B_t$, where $B.$ is the standard Brownian motion.

[*Hint: Consider, for each* $\varepsilon > 0$, *the Lévy process with characteristic exponent*

$$\psi_\varepsilon(u) = \int_{|y|\geq\varepsilon} \left(e^{iuy} - 1 - iuy\mathbf{1}_{|y|<1}\right) K(dy).]$$

2 State carefully, without proof, the optional stopping theorem.

Let T be a stopping time with $\mathsf{E}T < \infty$ and let X_n be a supermartingale with uniformly bounded increments, i.e., there exists a finite constant $K > 0$ such that

$$\left|X_n(\omega) - X_{n-1}(\omega)\right| \leq K \qquad \forall(n,\omega).$$

Show that X_T is integrable and $\mathsf{E}(X_T) \leq \mathsf{E}(X_0)$.

Consider successive flips of a coin having probability p of landing heads. Use a martingale argument to compute the expected number of flips until the sequence HHTTHHT appears.

3 Let B_t be a Brownian motion starting at the origin, $B_0 = 0$. Show that $M_t = \exp\{\lambda B_t - \frac{1}{2}\lambda^2 t\}$ is a martingale.

Consider the process $X_t = B_t + \mu t$, a Brownian motion with drift $\mu > 0$, $X_0 = 0$. For positive a and b, define the stopping time

$$T = \inf\{t \geq 0 : X_t = a \text{ or } X_t = -b\}.$$

Show that $T < \infty$ almost surely. Compute $\mathsf{P}(X_T = a)$ and $\mathsf{E}(T)$.

4 Define standard one-dimensional Brownian motion; state the Wiener theorem and sketch its proof.

Show that, almost surely,

(a) trajectories of the Brownian motion B_t are Hölder continuous of exponent α for all $\alpha < 1/2$;

(b) there is no interval (r, s) on which $t \mapsto B_t$ is Hölder continuous of exponent α for any $\alpha > 1/2$.

Explain briefly the relation of this result to differentiability properties of B_t.

5 Let μ and $(\mu_n : n \in \mathbb{N})$ be probability measures in $\mathbf{C}([0,1],\mathbb{R})$, the space of real continuous functions on $[0,1]$. It is known that if the sequence μ_n is tight and if *all* finite-dimensional distributions of μ_n converge weakly to those of μ, then the sequence μ_n converges weakly in $\mathbf{C}([0,1],\mathbb{R})$ to μ. Use this fact to get a proof of the Donsker invariance principle for random walks according to the following steps:

(a) Let $(S_n)_{n\geq 0}$ be a random walk with i.i.d. steps ξ of mean 0, variance 1 and finite fourth moment. Write $(S_t)_{t\geq 0}$ for the linear interpolation

$$S_{n+r} = (1-r)S_n + rS_{n+1}, \qquad r \in [0,1],$$

and denote by μ_N the probability distribution of $S_t^N = N^{-1/2} S_{Nt}$, $t \in [0,1]$. For any $k \geq 1$ and $0 = t_0 < t_1 < \ldots < t_k \leq 1$, let $\mu_N^{t_1,\ldots,t_k}$ be the law of

$$\big(S_{t_1}^N, S_{t_2}^N, \ldots, S_{t_k}^N\big).$$

Show that the characteristic functions

$$\varphi_{t_1,\ldots,t_k}^N(\lambda_1,\ldots,\lambda_k) = \mathsf{E}\exp\Big\{ i \sum_{l=1}^k \lambda_l \big(S_{t_l}^N - S_{t_{l-1}}^N\big)\Big\}$$

satisfy

$$\lim_{N\to\infty} \varphi_{t_1,\ldots,t_k}^N(\lambda_1,\ldots,\lambda_k) = \exp\Big\{ -\tfrac{1}{2} \sum_{l=1}^k {\lambda_l}^2 (t_l - t_{l-1})\Big\}$$

and thus deduce that all finite-dimensional distributions $\mu_N^{t_1,\ldots,t_k}$ converge to the corresponding finite-dimensional distributions of the Wiener measure μ on $[0,1]$.

(b) For every $N \geq 1$, consider a random process X_t^N such that

$$X_\bullet^N \in \mathbf{C_0}\big([0,1],\mathbb{R}\big) = \Big\{ f \in \mathbf{C}\big([0,1],\mathbb{R}\big) : f(0) = 0\Big\}$$

and let ν_N be its distribution. The sequence of such measures $(\nu_N ; N \geq 1)$ is known to be tight if for some positive C, γ, α and all $N \geq 1$, t_1, $t_2 \in [0,1]$,

$$\mathsf{E}\big|X_{t_1}^N - X_{t_2}^N\big|^\gamma \leq C|t_1 - t_2|^{1+\alpha}.$$

Verify that the sequence of measures μ_N in (a) is tight and thus deduce the Donsker invariance principle.

6 Let $\big(\xi_t : t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\big)$ be the simple symmetric random walk on $\mathbb{Z}^d$, $d \geq 3$, starting at the origin, $\xi_0 = 0$. Let, further, $\big(h(t,x) : t \in \mathbb{N}, x \in \mathbb{Z}^d\big)$ be i.i.d. random variables such that $\mathsf{P}(h = \pm 1) = 1/2$. For a fixed $\varepsilon \in (0,1)$ and $T \in \mathbb{N}$, consider the process

$$\kappa_T = \prod_{j=1}^{T} \big(1 + \varepsilon h(j, \xi_j)\big).$$

Denote by $\langle \cdot \rangle$ the expectation w.r.t. the process ξ and by $\mathsf{E}(\cdot)$ the expectation w.r.t. the h-variables. The aim is to describe the limiting behaviour of $\langle \kappa_T \rangle$ as $T \to \infty$.

(a) Verify that

$$\langle \kappa_t \rangle \equiv \frac{1}{(2d)^t} \sum_{\omega} \prod_{j=1}^{t} \big(1 + \varepsilon h(j, \xi_j)\big)$$

is a martingale w.r.t. the filtration $\big(\mathcal{F}_t\big)_{t \geq 0}$, $\mathcal{F}_t = \sigma\big(h(s,x) : s \leq t, x \in \mathbb{Z}^d\big)$. The summation above is over all nearest neighbour paths $\omega = (\xi_1, \ldots, \xi_t)$ of length t starting at the origin. Deduce that $\big\langle \kappa_t \big\rangle$ converges a.s., as $t \to \infty$, to some random variable $\zeta \geq 0$.

(b) Let $\xi^{(1)}$, $\xi^{(2)}$ be two independent copies of the random walk ξ (independent also of the h-variables) with the corresponding processes

$$\kappa_t^{(i)} = \prod_{j=1}^{t} \big(1 + \varepsilon h(j, \xi_j^{(i)})\big), \quad i = 1, 2.$$

Using the identity

$$\mathsf{E}\big(\langle \kappa_t \rangle^2\big) = \mathsf{E}\big(\big\langle \kappa_t^{(1)} \kappa_t^{(2)} \big\rangle\big)$$

(and by expressing $\mathsf{E}\big(\big\langle \kappa_t^{(1)} \kappa_t^{(2)} \big\rangle\big)$ in terms of the intersections of $\xi^{(1)}$ and $\xi^{(2)}$) or otherwise, show that, for ε small enough, $\big\langle \kappa_t \big\rangle$ is a martingale bounded in L^2 and thus its limit ζ satisfies $\mathsf{E}\zeta = \mathsf{E}\big\langle \kappa_t \big\rangle = 1$.

[**Hint.** *You may use without proof the following transience property of random walks in* $\mathbb{Z}^d$ *whose steps are bounded and symmetrically distributed w.r.t. zero: in dimension* $d \geq 3$, *after each visit to the origin such a walk has positive probability of never returning back to the origin.*]

(c) Show that $\{\zeta = 0\}$ is measurable w.r.t. the tail σ-field $\mathcal{T}_\infty$,

$$\mathcal{T}_\infty = \bigcap_t \sigma\big(h(s,x) : s \geq t, x \in \mathbb{Z}^d\big),$$

and deduce that $\mathsf{P}(\zeta = 0) = 0$, i.e. $\zeta > 0$ a.s.