UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 1.30 to 4.30

PAPER 21

ADVANCED PROBABILITY

*Attempt **FOUR** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State Doob's upcrossing inequality and deduce the almost sure martingale convergence theorem.

Let $(X_n)_{n\in\mathbb{Z}^+}$ be a martingale and suppose that $X_n \to Y$ a.s. as $n \to \infty$. Is it possible that $X_0 = 1$ and $Y = 0$ a.s?

Is it possible that $\mathbb{E}\,|\,Y\,| = \infty$?

Justify your answers.

2

Let $(X_n)_{n\geqslant 0}$ be a Markov chain on the integers with non-zero transition probabilities:

$$p_{2n,2n-1} = q_0, \quad p_{2n,2n} = r_0, \quad p_{2n,2n+1} = p_0,$$
$$p_{2n+1,2n} = q_1, \quad p_{2n+1,2n+1} = r_1, \quad p_{2n+1,2n+2} = p_1$$

for all n, where $q_0 + r_0 + p_0 = q_1 + r_1 + p_1 = 1$, and $p_0, q_0, p_1, q_1 > 0$. For $\theta \in \mathbb{R}$ set

$$X_n^\theta = \begin{cases} X_n, & \text{if } X_n \text{ is even,} \\ X_n - \theta, & \text{if } X_n \text{ is odd.} \end{cases}$$

Show that $(X_n^\theta)_{n\geqslant 0}$ is a martingale for some θ if and only if

$$\frac{p_0 - q_0}{p_0 + q_0} + \frac{p_1 - q_1}{p_1 + q_1} = 0 \qquad (*)$$

Fix positive integers a and b. Set $X_n^{(N)} = X_n/N$ and

$$T_N = \inf\{n \geqslant 0 : \ X_n^{(N)} = -a \ \text{ or } X_n^{(N)} = b\}.$$

Let $\pi_N = \mathbb{P}_0\big(X_{T_N}^{(N)} = -a\big)$. Assuming that (*) holds, determine $\lim_{N\to\infty} \pi_N$ for all a and b.

What happens when (*) fails?

3 Show that, for any probability measure μ on $\mathbb{R}$, having characteristic function ϕ, for any $\lambda \in (0, \infty)$,

$$\mu\big(|y| \geqslant \lambda\big) \;\leqslant\; \big(1 - \sin 1\big)^{-1} \lambda \int_0^{1/\lambda} \big(1 - \mathrm{Re}\ \phi(u)\big)\ du.$$

Deduce that, for a sequence of such measures μ_n, with characteristic functions ϕ_n, the convergence of ϕ_n to the characteristic function ϕ implies tightness of the μ_n.

Let $X_1, X_2, \ldots$ be independent and identically distributed integrable random variables, having mean 0 and characteristic function ϕ. Show that, for some continuous function ψ with $\psi(0) = 0$,

$$|1 - \phi(u)^n| \leqslant \psi(u) n |u| \quad \text{for all} \quad u \in \mathbb{R}, n \geqslant 0$$

and deduce that

$$\frac{X_1 + \ldots + X_n}{n} \;\to\; 0 \quad \text{in probability.}$$

4 Prove or disprove each of the following statements about Brownian motion $(B_t)_{t \geqslant 0}$ in the limit $t \to \infty$:

(a) $B_t \to \infty$ a.s.,

(b) $B_t \not\to \infty$ a.s.,

(c) $B_t/t \to 0$ a.s.,

(d) $B_t/t \not\to 0$ a.s.,

(e) $B_t/\sqrt{t} \to 0$ a.s., (f) $B_t/\sqrt{t} \not\to 0$ a.s.

5 (a) Let $(X_t)_{t\geqslant 0}$ be a continuous non-negative submartingale. Fix $\lambda \geqslant 0$ and set

$$T = \inf\{t \geqslant 0 : \ X_t \geqslant \lambda\}.$$

Show carefully that T is a stopping time. Deduce that, for all t,

$$\lambda \mathbb{P}\big(X_t^* \geqslant \lambda\big) \ \leqslant \ \mathbb{E}\big(X_t 1_{X_t^* \geqslant \lambda}\big),$$

where $X_t^* \ = \sup\{X_s : s \leqslant t\}$, and hence that

$$\| \ X_t^* \ \|_2 \leqslant \ 2 \ \| \ X_t \ \|_2 \ .$$

(b) Let $(B_t)_{t\geqslant 0}$ be a Brownian motion. Compute $\mathbb{P}(\sup_{s\leqslant t} B_s > \lambda)$ for $\lambda \geqslant 0$ and hence show that

$$\| \sup_{s\leqslant t} B_s \ \|_2 \ = \ \sqrt{t}, \qquad t \geqslant 0.$$

Show, on the other hand, that

$$\sqrt{t} < \| \ \sup_{s\leqslant t} \ |B_s| \ \|_2 \leqslant 2\sqrt{t}, \qquad t \geqslant 0.$$

6

Suppose that $(X_t)_{t\geqslant 0}$ is a Lévy process and that both $(X_t)_{t\geqslant 0}$ and $(X_t^2 - t)_{t\geqslant 0}$ are martingales.

(a) If the paths of $(X_t)_{t\geqslant 0}$ are continuous, what is the distribution of X_1?

(b) If the paths of $(X_t)_{t\geqslant 0}$ move only by jumps of size ± 1, what is the distribution of X_1?

Justify your answers.

For positive integers a and b, set $T = \inf\{t \geqslant 0 : \quad X_t = -a \text{ or } X_t = b\}$.

Determine in both case (a) and case (b) the distribution of X_T.