UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 9.00 to 12.00

PAPER 41

ADVANCED FINANCIAL MODELS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathbf{A}_0$ be a fixed vector in $\mathbb{R}^r$ and $\mathbf{A}_1$ a random vector taking values in $\mathbb{R}^r$. Prove that exactly one of the following alternatives (a) or (b) holds:

(a) there exists a vector $\mathbf{x} \in \mathbb{R}^r$ satisfying either

$$\begin{array}{ll} & (i) \quad \mathbf{x}^\top \mathbf{A}_0 \leqslant 0\,, \quad \mathbf{x}^\top \mathbf{A}_1 \geqslant 0 \quad \text{and} \quad \mathbb{P}\big(\mathbf{x}^\top \mathbf{A}_1 > 0\big) > 0 \\ \text{or} & (ii) \quad \mathbf{x}^\top \mathbf{A}_0 < 0 \quad \text{and} \quad \mathbf{x}^\top \mathbf{A}_1 \geqslant 0\,; \end{array}$$

(b) there exists a positive random variable ν, $\mathbb{P}\,(\nu > 0) \;=\; 1$, with $\mathbb{E}\,\|\nu \mathbf{A}_1\| < \infty$ such that $\mathbf{A}_0 \;=\; \mathbb{E}\,(\nu \mathbf{A}_1)$.

Explain the relevance of this result to the characterization of lack of arbitrage in a one-period financial model.

2

Write an essay on the binomial model operating over n time periods. Your account should cover the following points:

(i) hedging and pricing contingent claims;

(ii) calculation of the Radon–Nikodym derivative for the change of probabilities;

(iii) the problem of maximizing expected utility for an investor within the model; and

(iv) the approximation of the Black–Scholes model by binomial models.

3

By using the Reflection Principle and Girsanov's Theorem, or otherwise, show that

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant t}(W_s+\nu s)\leqslant a\right) = \Phi\left(\frac{a-\nu t}{\sqrt{t}}\right) - e^{2a\nu}\Phi\left(\frac{-a-\nu t}{\sqrt{t}}\right), \quad \text{for } a > 0,$$

where $\{W_t,\, t \geqslant 0\}$ is a standard Brownian motion and Φ is the standard normal distribution function.

Hence, derive an expression for $\mathbb{E}\left[e^{\theta W_t} I_{(T_{a,b}>t)}\right]$, for any real θ, where $T_{a,b}$ is the hitting time of the line $a+bs$.

Determine an expression for $\mathbb{E}\left[e^{\theta T_{a,b}} I_{(T_{a,b}\leqslant t)}\right]$, for $2\theta \leqslant b^2$.

An American digital call pays £1 at the instant the stock price $\{S_t\}$ first reaches the level $c > S_0$, if this occurs during the lifetime $[0, t_0]$ of the option, otherwise it pays nothing. Explain how the result that you have obtained may be used to determine the time-0 price of this option in the context of the Black–Scholes model.

4

Explain what is meant by a *self-financing* portfolio in the Black–Scholes model. Suppose that the value of a portfolio at time t is a function of the stock-price process $\{S_t,\ t \geqslant 0\}$ and is given by

$$p(S_t,t) = g(S_t,t)\,S_t + h(S_t,t)\,e^{-\rho(t_0-t)},$$

where $g(x,t)$ and $h(x,t)$ are suitably smooth functions and ρ is the interest rate. Prove that this portfolio is self-financing on the time interval $[0,t_0]$ if and only if the equations

$$\begin{aligned} x\frac{\partial g}{\partial x} + e^{-\rho(t_0-t)}\frac{\partial h}{\partial x} &= 0, \quad \text{and} \\ \frac{1}{2}\sigma^2 x^2\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial t} + e^{-\rho(t_0-t)}\frac{\partial h}{\partial t} &= 0 \end{aligned}$$

are satisfied for $0 \leqslant t \leqslant t_0$, where σ is the volatility.

Deduce that the portfolio with value p is self-financing if and only if the function p satisfies the Black–Scholes equation.

Explain what changes would be necessary to the Black–Scholes equation for a claim which pays the holder at the rate $k(S_t)$ per unit time up until the expiry time t_0, for some function k.

5

In the context of the Black–Scholes model, explain how to determine the time-t price of a claim paying $f(S_{t_0})$ at time t_0, where $t < t_0$, for some function $f(\cdot)$ where S_t is the price of the stock at time t. Determine an explicit expression for the price in the case when $f(x) = x \ln x$.

Show that when the function f is convex then the holding in stock in the hedging (self-financing) portfolio rises when the stock price increases.

Let $p(\rho, \sigma)$ be the time-0 price considered as a function of the interest rate ρ and the volatility σ. Suppose that ρ is random with the $N(\rho_0, \tau^2)$ distribution, while σ and the other parameters determining the price are fixed. Show that

$$\mathbb{E}p(\rho, \sigma) = p\left(\rho_0 - \tau^2 t_0/2,\ \sqrt{\sigma^2 + \tau^2 t_0}\right).$$

6

Give a short description of the modelling of bond prices in terms of instantaneous forward interest rates and show that the discounted bond prices are martingales if and only if the bond prices $\{P_{s,t},\ 0 \leqslant s \leqslant t\}$ may be represented in terms of the short-rate process $\{R_s,\ s \geqslant 0\}$ as

$$P_{s,t} = \mathbb{E}\left[e^{-\int_s^t R_u\, du} \,\middle|\, \mathcal{F}_s\right], \quad \textit{for all} \quad 0 \leqslant s \leqslant t.$$

[*You should explain carefully all the terminology and notation used.*]

Now consider the one-factor model where

$$dR_s = \theta\, ds + \sigma\, dW_s$$

for constants θ and σ and $\{W_s,\ s \geqslant 0\}$ is a standard Brownian motion. Assume that $\mathcal{F}_s$ is generated by $\{W_u,\ 0 \leqslant u \leqslant s\}$ or, equivalently, by $\{R_u,\ 0 \leqslant u \leqslant s\}$. Show that when the discounted bond prices are martingales then the bond prices take the form

$$P_{s,t} = \exp\left[a_{s,t} - b_{s,t} R_s\right],$$

for suitable deterministic $a_{s,t}$ and $b_{s,t}$, which should be calculated explicitly.

Show that if the constant θ is replaced by a suitably-chosen deterministic function θ_s then this model may be made to fit any observed bond prices at time $s = 0$.

END OF PAPER