UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June 2006 1.30 to 4.30

PAPER 39

ADVANCED FINANCIAL MODELS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on optimal hedging in the least-squares sense in a one-period financial model. Your essay should cover the notions of attainable claims, dominated and equivalent martingale measures, the minimal martingale measure and a proof of the fact that the model is complete if and only if there is a unique dominated martingale measure.

2 Consider the standard binomial model operating over the times $0, 1, \ldots, n$ $(n \geqslant 2)$ where the stock price at time r is denoted by S_r. Let $g_r(S_r)$ represent the price at time r of a claim which pays $f(S_n)$ at time n. When f is convex show that g_r is convex on the possible values that S_r can take on (viz. $S_r = S_0 u^i d^{r-i}$, $i = 0, 1, \ldots, r$).

Show that when f is convex then the amount of stock held in the hedging portfolio increases between the times r and $r+1$ $(< n)$ if the stock price increases between r and $r+1$.

Now consider an investor who has initial wealth $w_0 > 0$, at time 0, and utility function $v(x) = \gamma x^{1/\gamma}$, for $x > 0$, where $\gamma > 1$; determine the claim that he would purchase in order to maximize the expected utility of his final wealth.

3 Let $T_{a,b}$ denote the first hitting time of the line $a + bs$ by a standard Brownian motion, where $a > 0$ and $-\infty < b < \infty$ and let $T_a = T_{a,0}$ represent the first hitting time of the level a.

For $\theta > 0$, using the fact that $\mathbb{E}\left(e^{-\theta T_a}\right) = e^{-a\sqrt{2\theta}}$ or otherwise, derive an expression for $\mathbb{E}\left(e^{-\theta T_{a,b}}\right)$ for each b, $-\infty < b < \infty$.

Hence, or otherwise, show that, for $t > 0$,

$$\mathbb{P}\left(T_{a,b} \leqslant t\right) = e^{-2ab}\Phi\left(\frac{bt-a}{\sqrt{t}}\right) + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right),$$

where Φ is the standard normal distribution function.

Use this result, in the context of the Black–Scholes model, to derive the price at time 0 of a barrier digital put which pays 1 at time t_0 if and only if the stock price stays below a predetermined barrier $c > S_0$ between times 0 and t_0, where S_0 is the initial price of the stock.

4 Suppose that in the Black–Scholes model, the stock price at time t is S_t, the fixed interest rate is ρ and the volatility is σ. Let $p(S_t, t)$ be the price at time t of a claim paying $C = f(S_{t_0})$ at time t_0; explain carefully why the function $p = p(x, t)$ satisfies the Black–Scholes equation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} + \rho x \frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} - \rho p = 0.$$

Now suppose that in addition to paying C at time t_0 the claim pays a dividend at rate $R_t = k(S_t, t)$ at time t. Explain how the Black–Scholes equation for the price of the claim $p(S_t, t)$ should be modified in this case. Justify your answer carefully.

5 For the Black–Scholes model, give a description of the pricing of a terminal-value claim paying the amount $f\left(S_{t_0}\right)$ at time t_0, where $\{S_t,\ t \geqslant 0\}$ is the stock price process. You may assume that f is a twice-differentiable function and your account should include a verification that the price satisfies the Black–Scholes equation as well as an analysis of its dependence on the various parameters of the model.

In particular, show that if f is convex and the replicating portfolio is short in bonds (that is, it holds a negative amount) then the price is a decreasing function of time.

6 Write an essay on modelling interest rates with Gaussian random fields. You need not include detailed proofs of results but you should outline how they are obtained.

END OF PAPER