UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Monday 3 June 2002   9 to 12

PAPER 29

ADVANCED FINANCIAL MODELS

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a one-period model in which there are $s + 1$ assets of which exactly one is riskless and the remainder are risky. Define (i) an *arbitrage* and (ii) an *equivalent martingale probability.*

Prove that there is no arbitrage if and only if there exists an equivalent martingale probability.

Suppose now that the underlying probability space is finite and that there is no arbitrage. Prove that the market is complete if and only if there is a unique equivalent martingale probability.

[*You may quote the Separating Hyperplane Theorem without proof but any other result that you use should be proved carefully.*]

2 Give a description of the standard binomial model operating over periods $r = 0, 1, \ldots, n$. Explain carefully how contingent claims are priced within the model.

Derive an expression in terms of the stock price S_n at time n for the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ of the martingale probability $\mathbb{Q}$ with respect to the original probability $\mathbb{P}$.

Consider an investor with initial wealth w_0 at time 0 who wishes to trade in the model so as to maximize the expected utility of his final wealth at time n, when his utility function is $v(x) = \left(1 - e^{-ax}\right)/a$, where $a > 0$ is a constant. Determine his optimal final wealth.

3 Let $\{W_t^\nu,\ t \geqslant 0\}$ denote a standard Brownian motion with drift ν and let $M_t^\nu = \sup_{0 \leqslant s \leqslant t} W_s^\nu$. By using the Reflection Principle and Girsanov's Theorem, or otherwise, prove that for $a > 0$ and $x \leqslant a$,

$$\mathbb{P}\left(W_t^\nu \leqslant x,\ M_t^\nu < a\right) = \Phi\left(\frac{x - \nu t}{\sqrt{t}}\right) - e^{2a\nu}\Phi\left(\frac{x - 2a - \nu t}{\sqrt{t}}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black–Scholes model, consider an up-and-in claim that pays $f\left(S_{t_0}\right)$ at time t_0 if a barrier $b > S_0$ is reached by the stock-price process $\{S_t,\ t \geqslant 0\}$ during the lifetime $[0, t_0]$ of the claim; otherwise it pays nothing. Prove that the price at time 0 of this claim is the same as that of an ordinary terminal-value claim, paying $g\left(S_{t_0}\right)$ at t_0, where

$$g(x) = f(x) I_{(x > b)} + (1/\kappa)^{\nu/\sigma} f(x/\kappa) I_{(x \leqslant \kappa b)},$$

and ν and κ are constants which should be specified.

4 Explain what is meant by a *self-financing* portfolio in the Black-Scholes model. Suppose that the value of a portfolio at time t is a function of the stock-price process $\{S_t,\ t \geqslant 0\}$ and is given by

$$f(S_t, t) = g(S_t, t)\, S_t + h(S_t, t)\, e^{-\rho(t_0 - t)},$$

where $g(x,t)$ and $h(x,t)$ are suitably smooth functions and ρ is the interest rate. Prove that this portfolio is self-financing if and only if the equations

$$x\frac{\partial g}{\partial x} + e^{-\rho(t_0-t)}\frac{\partial h}{\partial x} = 0, \quad \text{and}$$
$$\frac{1}{2}\sigma^2 x^2 \frac{\partial g}{\partial x} + x\frac{\partial g}{\partial t} + e^{-\rho(t_0-t)}\frac{\partial h}{\partial t} = 0$$

are satisfied for $0 \leqslant t \leqslant t_0$, where σ is the volatility.

Determine the self-financing portfolio, with initial value w_0 at time 0, that always maintains a fixed amount θ in 'real' terms in the holding in stock; that is, the value of the holding in stock always equals a fixed number, θ, of bonds.

5 For the Black–Scholes model, give a description of the pricing of a terminal-value claim paying the amount $f(S_{t_0})$ at time t_0, where $\{S_t,\ t \geqslant 0\}$ is the stock price process. You may assume that f is a twice-differentiable function and your account should include a verification that the price satisfies the Black–Scholes equation as well as an analysis of its dependence on the various parameters of the model.

In particular, show that if f is convex and the replicating portfolio is short in bonds (that is, it holds a negative amount) then the price is a decreasing function of time, while if f is concave and the replicating portfolio is long in bonds (that is, it holds a positive amount) then the price is an increasing function of time.

6 Let $P_{s,t} = e^{-\int_s^t F_{s,u} du}$ represent the price at time s of a bond paying 1 unit at time $t \geqslant s$. Suppose that the random variables $\{F_{s,t}\}$ are jointly Gaussian with $\mathbb{E}F_{s,t} = \mu_{s,t}$ and $\mathbb{C}\text{ov}\left(F_{s_1,t_1}, F_{s_2,t_2}\right) = c(s_1 \wedge s_2, t_1, t_2)$, for some appropriate function c with $c(0, t_1, t_2) \equiv 0$. Let $Z_{s,t}$ represent the discounted bond price so that $Z_{s,t} = e^{-\int_0^s R_u du} P_{s,t}$ where $R_u = F_{u,u}$ is the spot rate, and let $\mathcal{F}_s = \sigma\{F_{u,v} :\ 0 \leqslant u \leqslant s,\ u \leqslant v\}$ be the information at time s. Prove that the following three statements (a), (b) and (c) are equivalent:

(a) for each $t \geqslant 0$, $\{Z_{s,t}, \mathcal{F}_s, 0 \leqslant s \leqslant t\}$ is a martingale;

(b) $\mu_{s,t} = \mu_{0,t} + \displaystyle\int_0^t c(s \wedge v, v, t)dv$, for all (s,t), $0 \leqslant s \leqslant t$;

(c) $P_{s,t} = \mathbb{E}\left[e^{-\int_s^t R_u du} \,\middle|\, \mathcal{F}_s\right]$, for all (s,t), $0 \leqslant s \leqslant t$.