UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 1.30 to 4.30

PAPER 64

ADVANCED COSMOLOGY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The action for a point particle of mass m in a background metric $g_{\mu\nu}$ is

$$S = -\frac{m}{2}\int g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}d\tau\,,$$

where $x^{\mu} = (t, \mathbf{x})$, dots denote τ derivatives, and the canonical four-momentum $P^c_\mu \equiv \frac{\delta S}{\delta \dot{x}^\mu}$ is subject to the constraint

$$g^{\mu\nu}P^c_\mu P^c_\nu = -m^2\,.$$

For a flat FRW background, $g_{\mu\nu} = a^2(t)\eta_{\mu\nu}$, show that

(i) the canonical three-momentum $\mathbf{P}^c$ is conserved,

(ii) the special relativistic physical momentum $\mathbf{P} \equiv \gamma m\mathbf{v}$, with $\mathbf{v} = (d\mathbf{x}/dt)$ and $\gamma = 1/\sqrt{1-\mathbf{v}^2}$, decays as $a(t)^{-1}$.

Now consider the action for a massive scalar field,

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - \frac{1}{2}m^2\phi^2\right).$$

(iii) Compute S for a flat FRW background metric $g_{\mu\nu} = a^2(t)\eta_{\mu\nu}$ and show that the field equations read

$$\phi'' + 2\frac{a'}{a}\phi' - \boldsymbol{\nabla}^2\phi = -a^2m^2\phi\,, \qquad (*)$$

where primes denote $\frac{\partial}{\partial t}$.

(iv) For massless field, $m = 0$, in a radiation-dominated universe, $a \propto t$, show the general solution to $(*)$ may be written

$$\phi = \frac{1}{a(t)}\sum_{\mathbf{k}}(A_{\mathbf{k}}e^{i\mathbf{k}.\mathbf{x}}e^{-iwt} + c.c.)\,,$$

where $w = |\mathbf{k}|$ and $A_{\mathbf{k}}$ is constant for each $\mathbf{k}$.

(v) Recalling the quantisation rule $[P^c_i, x_j] = -i\hbar\delta_{ij} \to \mathbf{P}^c = \frac{\hbar}{i}\boldsymbol{\nabla}$, do the $\mathbf{k}$ modes of ϕ have fixed canonical, or physical momentum? How does this accord with your answers to parts (i) and (ii)?

2 In the 3+1 formalism for general relativity, the extrinsic curvature of constant-time slices Σ is given by

$$K_{\mu\nu} = P^{\alpha}_{\mu} P^{\beta}_{\nu} \nabla_{\alpha} n_{\beta} \,,$$

where the projector $P^{\nu}_{\mu} = \delta^{\nu}_{\mu} + n_{\mu} n^{\nu}$, and n^{μ} is the future-pointing unit normal to Σ, obeying $g_{\mu\nu} n^{\mu} n^{\nu} = -1$.

(i) Calculate n^{μ} and $K_{\mu\nu}$ explicitly for a flat FRW universe with line element

$$-dt^2 + a^2(t) d\mathbf{x}^2 \,,$$

and explain the meaning of the trace $K = g^{\mu\nu} K_{\mu\nu} = g^{ij} K_{ij}$.

(ii) The derivative operator D_{μ} on Σ is defined by taking the 3+1 covariant derivative ∇_{μ} and then projecting all indices onto Σ using the projector P^{ν}_{μ}.

Show that D_{μ} satisfies $D_{\mu}(g^{(3)}_{\alpha\beta}) = 0$, where $g^{(3)}_{\alpha\beta} = g_{\alpha\beta} + n_{\alpha} n_{\beta}$ is the induced three-metric on Σ.

(iii) Explain why $K_{\mu\nu}$ is symmetric.

(iv) Starting from the Gauss relation

$$R^{(3)\lambda}{}_{\gamma\nu\mu} = P^{\alpha}_{\mu} P^{\epsilon}_{\nu} P^{\xi}_{\gamma} P^{\lambda}_{\eta} R^{(4)\eta}{}_{\xi\alpha\epsilon} - K_{\mu\gamma} K^{\lambda}{}_{\nu} + K_{\nu\gamma} K^{\lambda}{}_{\mu} \,,$$

derive the identity

$$G^{(4)}_{\mu\nu} n^{\mu} n^{\nu} = \frac{1}{2} (R^{(3)} + K^2 - K_{\mu\nu} K^{\mu\nu}) \,.$$

For a flat FRW background, which of Einstein's equations does this correspond to, and what is the meaning of each term on the right hand side?

3 In synchronous gauge for linear perturbations about a flat ($k = 0$) FRW background, the metric is taken to be

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] ,$$

where $|\det(h_{ij})| \ll 1$ and the relevant connections are $\Gamma^0_{00} = a'/a$, $\Gamma^0_{0i} = \Gamma^i_{00} = 0$,

$$\Gamma^0_{ij} = \frac{a'}{a}\left[\delta_{ij} + h_{ij}\right] + \frac{1}{2}h'_{ij}\,, \qquad \Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} + \frac{1}{2}h'_{ij}\,, \qquad \Gamma^i_{jk} = \frac{1}{2}[h_{ij,k} + h_{ik,j} - h_{jk,i}]\,,$$

with primes (e.g. a') denoting differentiation with respect to the conformal time τ.

(i) Assume the universe is filled with a perfect fluid with energy-momentum tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}\,,$$

where u^μ is the fluid 4-velocity, ρ is the energy density and P is the pressure, with the latter related through an equation of state $P = w\rho$. In a comoving synchronous frame (i.e. $u^\mu = a^{-1}(1,\ \mathbf{v})$ with $|\mathbf{v}| \ll 1$), *linearise* the energy-momentum tensor to find that in Fourier space we have

$$T^{00} = \frac{1}{a^2}\bar\rho(1+\delta)\,, \qquad T^{0i} = \frac{1}{a^2}(1+w)\bar\rho\, ik_i\theta\,, \qquad T^{ij} = \frac{1}{a^2}w\bar\rho\left[(1+\delta)\delta_{ij} - h_{ij}\right]\,,$$

with k_i the components of the comoving wavevector $\mathbf{k}$, the background density $\bar\rho = \bar\rho(t)$, and where suitable definitions should be given for the density perturbation δ and the velocity potential θ. Show that energy-momentum conservation equation $T^{0\mu}{}_{;\mu} = 0$ yields

$$\delta' - (1+w)k^2\theta + \tfrac{1}{2}(1+w)h' = 0\,,$$

where $h = h_{ii}$. Discuss the relation of a cold dark matter perturbation δ_c (i.e. $P = 0$ and comoving $\mathbf{v} = 0$) to the metric perturbation h_{ij}, demonstrating that density change is due to volume change in synchronous gauge.

(ii) Now assume that the cold dark matter density perturbation evolution equation is given by

$$\delta''_c + \frac{a'}{a}\delta'_c - \frac{3}{2}\left(\frac{a'}{a}\right)^2 \Omega_c\, \delta_c = 0\,,$$

where the density parameter is $\Omega \equiv (8\pi G/3)\bar\rho(a/a')^2$. Why is this equation only valid during the matter-dominated era (when $a \propto \tau^2$), that is, only after equal-matter radiation ($\tau > \tau_{eq}$) and before the accelerating phase ($\tau < \tau_\Lambda$)? Find the growing and decaying mode solutions for δ_c while $\tau_{eq} < \tau < \tau_\Lambda$.

During the radiation era ($\tau < \tau_{eq}$), adiabatic cold dark matter perturbations on superhorizon scales ($k < 2\pi/\tau$) grow as $\delta_c \propto \tau^2$, but subhorizon ($k > 2\pi/\tau$) perturbations stagnate, $\delta_c \approx$ const. Provide brief physical explanations for these different behaviours.

Use these growing mode solutions to derive an approximate transfer function $T(k)$ in Fourier space to project forward to today a primordial set of initial perturbations, that is,

$$\delta_{\mathbf{k}}(\tau_0) = T(k)\delta_{\mathbf{k}}(\tau_i)\,,$$

where $k = |\mathbf{k}|$, initially $\tau_{\mathrm{i}} \ll \tau_{\mathrm{eq}}$ and, for simplicity, assume that today $\tau_0 \approx \tau_\Lambda$ (ignore acceleration). For a primordial scale-invariant power spectrum $P_{\mathrm{i}}(k) = Ak$ (A is a constant), use this transfer function to find the power spectrum today. Comment on any significant features in the power spectrum and the lengthscales and observed structures with which they might be associated.

4 Ignoring metric perturbations, a moment expansion of the collisional Boltzmann equation yields equations for the photon density perturbation δ_γ, velocity potential θ_γ and shear viscosity σ_γ (here terminating at $\ell = 2$):

$$\delta'_\gamma - \tfrac{4}{3}k^2\theta_\gamma = 0\,, \qquad \theta'_\gamma + \tfrac{1}{4}\delta_\gamma - \sigma_\gamma = -an_{\rm e}\sigma_{\rm T}(\theta_\gamma - \theta_{\rm b})/k^2\,,$$

$$\sigma'_\gamma + \frac{4}{15}k^2\theta_\gamma = -an_{\rm e}\sigma_{\rm T}\sigma_\gamma\,.$$

where k is the wavenumber, σ_T is the Thomson cross-section and $n_{\rm e}$ is the electron density.

The corresponding equations for the coupled baryons are

$$\delta_b - k^2\theta_b = 0\,, \qquad \theta'_{\rm b} + \tfrac{a'}{a}\theta_{\rm b} = -Ran_{\rm e}\sigma_{\rm T}(\theta_{\rm b} - \theta_\gamma)/k^2\,,$$

where $c_{\rm s}$ is the sound speed and R is given by the ratio of background photon and baryon densities $R = (4/3)\bar\rho_\gamma/\bar\rho_{\rm b}$.

(i) For initially adiabatic perturbations we have $\delta_\gamma \approx \frac{4}{3}\delta_{\rm b}$; explain why. While the baryons and photons are tightly coupled we will also have $\theta_\gamma \approx \theta_{\rm b}$. Show that in this limit, the photon and baryon evolution equations can be combined to become

$$\delta'_\gamma = \frac{4}{3}k^2\theta_\gamma\,, \qquad \theta'_\gamma = -3c_{\rm s}^2\left[\frac{1}{4}\delta_\gamma - \sigma_\gamma + \frac{1}{R}\frac{a}{a'}\theta_\gamma\right]$$

where the effective sound speed is given by $c_{\rm s}^2 \equiv \frac{1}{3}R/(1+R) = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\bar\rho_{\rm b}}{\bar\rho_\gamma}\right)^{-1}$.

(ii) In the limit that decoupling is in the matter era (cold dark matter $\Omega_{\rm c} \approx 1$ and $a \propto \tau^2$) with $\bar\rho_\gamma \gg \bar\rho_{\rm b}$, show that the tight coupling equations can be combined to become

$$\delta''_\gamma - \frac{4}{3}k^2\sigma_\gamma + \frac{1}{3}k^2\delta_\gamma = 0\,.$$

Assuming that $\sigma'_\gamma \approx 0$, show that this yields

$$\delta''_\gamma + \frac{4}{15}\tau_{\rm c}k^2\delta'_\gamma + \frac{1}{3}k^2\delta_\gamma = 0\,.$$

where $\tau_{\rm c} = (a\sigma_{\rm T}n_{\rm e})^{-1}$. Find general solutions of this equation (ignoring terms of $\mathcal{O}(\tau'_{\rm c}, \tau_{\rm c}^2)$) and show that for subhorizon scales they take the following approximate form

$$\delta_\gamma(\mathbf{k},\tau) \approx \Big[A(\mathbf{k})\cos(k\tau/\sqrt{3}) + B(\mathbf{k})\sin(k\tau/\sqrt{3})\Big]\,\exp(-k^2/k_{\rm D}^2)\,,$$

where $k_{\rm D}$ is a time-dependent damping scale.

[*Hint: Consider the substitution* $\delta_\gamma = \tilde\delta_\gamma \exp\left(-\dfrac{2k^2}{15}\displaystyle\int_0^\tau \tau_{\rm c}d\tau\right)$.]

(iii) Temperature anisotropies in the cosmic microwave background are given in synchronous gauge by

$$\frac{\Delta T}{T}(\mathbf{x},\hat{\mathbf{n}},\tau_0) = \frac{1}{4}\delta_\gamma(\mathbf{x},\tau_{\rm dec}) + \hat{\mathbf{n}}\cdot\mathbf{v}_\gamma(\mathbf{x},\tau_{\rm dec}) - \frac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} h'_{ij}\hat n^i\hat n^j d\tau\,.$$

Consider the variance of the solution obtained in (ii) to discuss the effect on the angular power spectrum of intrinsic photon density fluctuations and Doppler motions. Qualitatively compare and contrast the influence of metric perturbations and the relative lengthscales on which they are important.

5 Consider a scalar field ϕ with action

$$S = \int d^4x\sqrt{-g}(-\frac{1}{2}(\partial\phi)^2 - V(\phi))\,.$$

(i) Show this yields a stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right]\,.$$

[*Hint: You may use* $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\delta g_{\alpha\beta}g^{\alpha\beta}$ *and* $\delta(g^{\alpha\beta}) = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$.]

(ii) Compute the energy density $\rho = -T^0_0$ and pressure $P = \frac{1}{3}T^i_i$ due to a homogeneous scalar field $\bar{\phi}(t)$ in a flat FRW background with line element

$$a^2(t)(-dt^2 + d\mathbf{x}^2)\,,$$

where t is the conformal time.

(iii) If ϕ is perturbed,

$$\phi = \bar{\phi}(t) + \delta\phi(t,\mathbf{x})\,,$$

show that the momentum density

$$\delta T^0_i = -\frac{1}{a^2}\bar{\phi}'\partial_i\delta\phi\,.$$

(iv) For a general scalar metric perturbation, the line element reads

$$ds^2 = a^2(t)\left[-(1+2\Phi)dt^2 - 2\partial_i\beta dx^i dt + \left[\delta_{ij}(1-2\Psi) + 2\partial_i\partial_j\chi\right]dx^i dx^j\right]\,.$$

Under a coordinate transformation $\tilde{x}^\mu = x^\mu + \xi^\mu$, we have

$$\begin{aligned}\tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) &= -g_{\mu\alpha}\xi^\alpha_{,\nu} - g_{\nu\alpha}\xi^\alpha_{,\mu} - g_{\mu\nu,\alpha}\xi^\alpha\,,\\ \tilde{\phi}(x) - \phi(x) &= -\xi^\alpha\partial_\alpha\phi\,.\end{aligned}$$

In particular, if $\xi^0 = T$ and $\xi^i = \partial_i L$ work out the transformation of Ψ and $\delta\phi$.

Write down a gauge-invariant combination of Ψ and $\delta\phi$ and give its physical interpretation.

END OF PAPER