UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2008 9.00 to 11.00

PAPER 43

ACTUARIAL STATISTICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let S be the total amount claimed in a year on a portfolio of home insurance policies, where the successive claim sizes $X_1, X_2, \ldots$ are independent identically distributed positive integer-valued random variables, independent of the number N of claims that arrive during the year. Let $p_n = \mathbb{P}(N = n)$, $n = 0, 1, 2, \ldots$, and assume that

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \quad n = 1, 2, \ldots, \tag{1}$$

for some known constants a and b. Assume that $f_k = \mathbb{P}(X_1 = k)$, $k = 1, 2, \ldots$, are known, and let $g_k = \mathbb{P}(S = k)$, $k = 0, 1, 2, \ldots$. Derive a recursion formula for the g_k's in terms of a, b and the f_k's.

Show that if (1) is satisfied then $a + b \geqslant 0$.

Assume that (1) is satisfied. Find the distribution of N in the following cases:

(a) $a + b = 0$;

(b) $a = 0$ and $b > 0$;

(c) $0 < a < 1$ and $b = (m - 1)a$ for some positive integer m.

2 Describe what is meant by *quota-share reinsurance*, *excess-of-loss reinsurance* and *stop-loss reinsurance.*

Claims arrive at a direct insurer in a Poisson process with rate λ per year. The claim sizes $X_1, X_2, \ldots$ are independent identically distributed random variables with finite mean μ and moment generating function $M_X(t)$, independent of the claim arrivals process. Assume that the premium loading factor θ is positive and that the direct insurer's initial capital is $u > 0$. Find the moment generating function $M_S(t) = \mathbb{E}(e^{St})$, where S is the total amount claimed during one year. Find the moment generating function of the total capital at the end of one year.

The direct insurer takes out a one-year quota-share reinsurance contract with retained proportion α, $0 < \alpha < 1$. Write down the premium charged by the reinsurer if the reinsurer's premium loading factor is $\theta_R > 0$. With this reinsurance contract, find the moment generating function of the direct insurer's total capital W at the end of one year.

The direct insurer chooses the value of α to maximise $\mathbb{E}(g(W))$ where $g(w) = -e^{-\beta W}$ for a fixed positive constant β. If the original claims size $X_1, X_2, \ldots$ have an exponential distribution with mean μ, find the optimal choice α_{opt} of α. Discuss briefly how α_{opt} depends on (a) θ_R and (b) θ.

3 In a classical risk model, the claims $X_1, X_2, \ldots$ have distribution function $F_X(x)$, density $f_X(x)$, mean μ and moment generating function $M_X(r)$. The initial capital is $u > 0$, and the safety loading θ is positive. Assume that there exists r_∞, $0 < r_\infty \leqslant \infty$, such that $M_X(r)$ increases to infinity as r tends to r_∞ from the left, and that $R, 0 < R < r_\infty$, satisfies

$$M_X(R) - 1 = (1+\theta)\mu R.$$

Define the *probability of ruin*, $\psi(u)$, with initial capital u. State the Lundberg inequality and write down the Cramér–Lundberg approximation for $\psi(u)$. Show that R satisfies $\frac{1}{\mu}\int_0^\infty e^{Rx}(1-F_X(x))dx = (1+\theta)$.

Suppose that F_X satisfies $1 - F_Y(x) \leqslant 1 - F_X(x)$ for all $x \geqslant 0$, where F_Y is the distribution function of a random variable Y with density $f_Y(x) = (1-F_X(x))/\mu$; in this case X_1 is said to be *new better than used in expectation* (NBUE). Show that $\mathbb{E}(e^{RY}) \leqslant \mathbb{E}(e^{RX_1})$, and hence show that

$$R \geqslant \frac{\theta}{(1+\theta)\mu}. \qquad (\star)$$

Suppose that $\theta = 0.1$ and that $f_X(x) = xe^{-x}$, $x > 0$. Find R. Show that X_1 is NBUE and verify that $(\star)$ holds.

4 Explain what is meant by a *credibility estimate*, a *credibility factor* and *Bayesian credibility.*

A portfolio of life insurance policies has m_i policies in year i, where $m_i \geqslant 1$ and m_i is known, $i = 1, 2, \ldots, k+1$. Let X_i be the number of claims on the portfolio in year i and assume that, given θ, the random variables $X_1, X_2, \ldots, X_{k+1}$ are conditionally independent with

$$\mathbb{P}(X_i = x) = \binom{m_i}{x}\theta^x(1-\theta)^{m_i - x}, \quad x = 0, 1, \ldots, m_i.$$

Suppose further that θ has prior density $f(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$, $0 < \theta < 1$, for known positive α and β. Suppose that the numbers of claims $x_1, \ldots, x_k$ in years 1 to k are observed and let $\mu(\theta) = \mathbb{E}(X_{k+1} \mid \theta)$.

Find the Bayesian estimate $\mathbb{E}(\mu(\theta) \mid X_1 = x_1, \ldots, X_k = x_k)$ of the expected number of claims in year $k+1$ under quadratic loss. Show that this estimate can be written as a credibility estimate, and write down the credibility factor Z. Discuss the behaviour of Z if

(a) k becomes large (with α and β fixed);

(b) α and β increase in such a way that $\alpha/(\alpha+\beta)$ remains constant (and k is fixed).

[Hint: *If Y has density proportional to* $y^{\alpha-1}(1-y)^{\beta-1}$, $0 < y < 1$, *then* $\mathbb{E}(Y) = \frac{\alpha}{\alpha+\beta}$ *and* $\mathrm{var(Y)} = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$.]

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