UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 13:30 to 15:30

PAPER 41

Actuarial Statistics

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) A portfolio consists of m independent risks. The total amount S_i claimed on risk i in one year has a compound Poisson distribution with Poisson parameter λ_i and claim size distribution function F_i, $i = 1, \ldots, m$. Let S be the total amount claimed on the whole portfolio in one year. Show that S has a compound Poisson distribution, and find the resulting Poisson parameter and claim size distribution function.

(ii) Let $N_1, \ldots, N_m$ be independent Poisson random variables with $\mathbb{E}(N_i) = \lambda_i$, and let $a_1, \ldots, a_m$ be distinct non-negative constants. Show that

$$T = a_1 N_1 + \ldots + a_m N_m$$

has a compound Poisson distribution, and find the corresponding Poisson parameter and claim size distribution.

(iii) Suppose that the claim sizes $X_1, X_2, \ldots$ for a particular risk are independent, identically distributed random variables taking values in $\{a_1, \ldots, a_m\}$ with $p_i = \mathbb{P}(X_1 = a_i)$. The number N of claims in a year for this risk has a Poisson distribution with mean λ and is independent of $X_1, X_2, \ldots$. Let $\tilde{T}$ be the total amount claimed for this risk in one year. Show that $\tilde{T}$ can be represented in the form

$$a_1 N_1 + \ldots + a_m N_m$$

where, given $N = n$, $(N_1, \ldots, N_m)$ has a multinomial distribution with parameters $n, p_1, \ldots, p_m$, so that

$$\mathbb{P}(N_1 = n_1, \ldots, N_m = n_m | N = n) = \frac{n!}{n_1! \ldots n_m!} \quad p_1^{n_1} \ldots p_m^{n_m}$$

if $n = \sum_{i=1}^{m} n_i$, and is zero otherwise.

Show that

$$\mathbb{E}\left[\prod_{i=1}^{m} z_i^{N_i}\right] = \prod_{i=1}^{m} \exp\{\lambda p_i (z_i - 1)\},$$

and write down the distribution of N_i.

[Hint: If $(Y_1, \ldots, Y_m)$ has a multinomial distribution with parameters $n, p_1, \ldots, p_m$ then $\mathbb{E}[\prod_{i=1}^{m} z_i^{Y_i}] = (p_1 z_1 + \ldots + p_m z_m)^n$]

2 The total amount S claimed on a portfolio of fire insurance policies in a year has a compound Poisson distribution with Poisson parameter λ and claim size density $f(x)$. Derive an expression for the cumulant generating function of S in terms of λ and the moment generating function of the claim size distribution. Hence show that the j^{th} cumulant of S is $\lambda\mathbb{E}[X^j]$, where X is an individual claim size.

The insurer takes out excess of loss reinsurance with retention M. Find the mean and variance of the total amounts S_I and S_R paid in a year by the insurer and reinsurer, respectively.

Suppose that $f(x) = \begin{cases} \dfrac{3d^3}{x^4} & x > d \\ 0 & x \leqslant d \end{cases}$

Sketch $\text{var}(S_I) + \text{var}(S_R)$ as a function of M, and find the value of M that minimises $\text{var}(S_I) + \text{var}(S_R)$ for this claim size distribution.

3 Consider a classical risk model where claims arrive in a Poisson process with rate λ and claims are independent, identically distributed random variables with distribution function F, moment generating function M and mean μ, independent of the arrivals process. The premium rate is c. Assume positive safety loading, and that there exists r_∞, $0 < r_\infty \leqslant \infty$, such that $M(r) \uparrow \infty$ as $r \uparrow r_\infty$. Define the adjustment coefficient, R.

Let $\psi(u)$ be the probability of ruin when the initial capital is u, and let $\phi(u) = 1 - \psi(u)$. You are given that $\phi(u)$ satisfies

$$\phi(u) = 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u \phi(u-x)(1-F(x))\,dx.$$

Show how to derive a renewal-type equation for $Z(u) = e^{Ru}\psi(u)$. Quoting without proof results from renewal theory as necessary, derive an expression for $A = \lim_{u\to\infty} Z(u)$.

Calculate A for a portfolio where the claim sizes have density

$$f(x) = xe^{-x} \qquad (x > 0).$$

For a different portfolio, it is found that

$$\psi(u) = ae^{-u} + b\,e^{-6u}$$

for constants a and b. Find R, A and $\dfrac{\lambda\mu}{c}$ for this portfolio.

4 Let $X_1, X_2, \ldots$ be independent Poisson random variables, each with mean θ,where the prior distribution of θ is gamma with mean $\frac{\alpha}{\beta}$ and variance $\frac{\alpha}{\beta^2}$.

Find $\mathbb{E}[X_{n+1}|X_1 = x_1, \ldots, X_n = x_n]$ and show that it can be written in the form of a credibility estimate.

A risk in year j consists of n_j independent policies and the number of claims on each policy has a Poisson distribution with mean θ, $j = 1, \ldots, n+1$, and θ has the prior distribution above. At $t = 0$, claim sizes are a fixed amount c. Claims inflation is $100r\%$ per year and claims are settled at the end of each year, so that claim sizes for year j are $(1+r)^j\, c$. Let Y_j denote the average amount claimed per policy in year j. Given $Y_1 = y_1, \ldots,$ $Y_n = y_n$, find the posterior distribution of θ. Find $\mathbb{E}[Y_{n+1}|Y_1 = y_1, \ldots, Y_n = y_n]$ and show that it can be written in the form

$$Z\, m(\mathbf{y}) + (1 - Z)\, m\ .$$

Give explicit expressions for Z, m and $m(\mathbf{y})$, and interpret m and $m(\mathbf{y})$ in words. Explain what happens to Z as the total past exposure $\displaystyle\sum_{j=1}^{n} n_j$ increases.