UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June, 2003 9:00 to 11:00

PAPER 39

Actuarial Statistics

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let N and X_1 be random variables taking values in $\{0, 1, 2, \ldots\}$ with $\mathbb{P}(X_1 = 0) > 0$. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, and let $S = \sum_{i=1}^{N} X_i$. Derive the probability generating function $G_S(z) = \mathbb{E}[z^S]$ in terms of the probability generating functions $G_N(z)$ and $G_X(z)$ of N and X_1 respectively.

Let $p_n = \mathbb{P}(N = n)$ and suppose $p_n = \left(a + \frac{b}{n}\right) p_{n-1}$, $n = 1, 2, \ldots$ for some constants a and b. Show that $G'_N(z) = \dfrac{a+b}{1-az} G_N(z)$ and that $G'_S(z) = aG'_S(z)G_X(z) + (a + b)G_S(z)G'_X(z)$. Let $\mathbb{P}(X_1 = k) = f_k$ and $\mathbb{P}(S = k) = g_k$, $k = 0, 1, 2, \ldots$. By equating coefficients of z^{r-1}, or otherwise, show that

$$g_r = \frac{1}{1 - af_0} \sum_{j=1}^{r} \left(a + \frac{bj}{r}\right) f_j\, g_{r-j}, \quad r = 1, 2, \ldots\ .$$

Write down an expression for $\mathbb{P}(S = 0)$ in terms of the p_n's and f_k's.

(ii) A portfolio covers claims for a type of severe accident for m independent factories. For each factory, the probability of exactly one such accident is $p\,(0 < p < 1)$ and the probability of more than one accident is negligible. Define $q = 1 - p$. The number of claims arising from an accident in a factory has a Poisson distribution with mean $\lambda\,(\lambda > 0)$. Find the probability that there are no claims for this portfolio in a particular year. Show that, following the notation given in (i), $p_n = \left(a + \dfrac{b}{n}\right) p_{n-1}$ with $a = -\dfrac{p}{q}$ and $b = \dfrac{(m+1)p}{q}$.

2 Claims $X_1, X_2, \ldots$ are independent and identically distributed, with density

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \quad \exp\left\{-\tfrac{1}{2}\left(\tfrac{\log x - \mu}{\sigma}\right)^2\right\}, \quad x > 0.$$

Show that

$$\int_a^b x^k\, f(x)\, dx = \exp\left\{k\mu + \frac{k^2\sigma^2}{2}\right\}\left[\Phi\Big(\frac{\log b - \mu - k\sigma^2}{\sigma}\Big) - \Phi\Big(\frac{\log a - \mu - k\sigma^2}{\sigma}\Big)\right]$$

where $0 \leqslant a < b \leqslant \infty$, and Φ is the standard normal distribution function.

Reinsurance is arranged for these claims, where the reinsurer pays the excess of any claim over M, up to a maximum reinsurance payment of A, where M and A are positive constants. Given that a claim exceeds M, show that the conditional expected value paid for that claim by the reinsurer is

$$\frac{1}{1 - \Phi\left(\frac{\log M - \mu}{\sigma}\right)} \quad \left[e^{\mu + \frac{\sigma^2}{2}}\left(\Phi\Big(\frac{\log(M+A) - \mu - \sigma^2}{\sigma}\Big) - \Phi\Big(\frac{\log M - \mu - \sigma^2}{\sigma}\Big)\right)\right.$$
$$- M\left(\Phi\Big(\frac{\log(M+A) - \mu}{\sigma}\Big) - \Phi\Big(\frac{\log M - \mu}{\sigma}\Big)\right)$$
$$\left. + A\left(1 - \Phi\Big(\frac{\log(M+A) - \mu}{\sigma}\Big)\right)\right].$$

If claims on these policies increase by 10% (but M and A stay the same), find the conditional expected amount paid by the reinsurer on a claim, given that the claim exceeds M.

3 A classical risk model has Poisson rate λ, relative safety loading $\rho > 0$ and mean claim size μ. Assume there exists r_∞, $0 < r_\infty \leqslant \infty$, such that $M_X(r) \uparrow \infty$ as $r \uparrow r_\infty$, where $M_X(r)$ is the claim size moment generating function.

(i) Define the probability of ruin and the adjustment coefficient. State the Lundberg inequality.

(ii) An insurer mistakenly assumes that the claims are exponentially distributed. Show that the adjustment coefficient calculated by the insurer is $R_I = \dfrac{\rho}{(1+\rho)\mu}$.

(iii) The true claim size density is $f(x) \propto x^{-1/2} e^{-\beta x}$ $(x > 0)$ where $\beta = \dfrac{1}{2\mu}$. Show that the true adjustment coefficient is

$$R = \beta\Big(\frac{\rho - 3 + \sqrt{(\rho+9)(\rho+1)}}{2(1+\rho)}\Big)\ .$$

Compare the corresponding bounds on the probability of ruin and comment.

4 Explain the terms *credibility premium* and *credibility factor*.

The total claim amount on a certain risk in year j is X_j, $j = 1, 2, \ldots$, where, given θ, $X_1, X_2, \ldots$ are conditionally independent and $X_j \sim N(\theta, \sigma_1^2)$, σ_1^2 known. *A priori* $\theta \sim N(\mu, \sigma_2^2)$, where μ and σ_2^2 are known. Show that, given $X_1, \ldots, X_n$, the Bayesian estimate of $\mathbb{E}[X_{n+1}|\theta]$ under quadratic loss can be written in the form of a credibility estimate and state the credibility factor.

Now suppose the risk consists of m_j lives in year j. Let the expectation and variance of the amount claimed on a single life in one year be θ and v respectively. As an approximation, suppose that the total claim amount Y_j arising from the whole risk (i.e., from all m_j lives) in year j is normally distributed. Define $X_j = Y_j/m_j$, for $1 \leqslant j \leqslant n+1$. Assume that $\theta \sim N(\mu, \sigma^2)$ *a priori* and that v, μ and σ^2 are known. The amounts $Y_1, \ldots, Y_n$ are observed and $m_1, \ldots, m_{n+1}$ are known.

Find the Bayesian estimate under quadratic loss of the premium per life for year $n+1$, and show that this can be written as a credibility estimate. Discuss what happens to the credibility factor if (i) σ^2 increases (ii) v increases. Write down the premium for the whole risk for year $n+1$.