UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 1.30 to 3.30

PAPER 82

ACOUSTICS

*Attempt **TWO** questions.*

*There are **FOUR** questions in total.*

The questions carry equation weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Lighthill's equation describing aerodynamic sound generation is

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \ , \tag{1}$$

where for an inviscid fluid $T_{ij} = \rho u_i u_j + p - c_0^2 \rho$ is the quadrupole distribution.

(i) Using equation (1), together with the free-space Green's function for the wave equation in three dimensions,

$$G(\mathbf{x}, t) = \frac{\delta(t - |\mathbf{x}|/c_0)}{4\pi |\mathbf{x}| c_0^2} \ ,$$

show that the far-field sound generated by a compact quadrupole distribution is

$$\rho'(\mathbf{x}, t) = \frac{x_i x_j \ddot{S}_{ij}(t - |\mathbf{x}|/c_0)}{4\pi c_0^4 |\mathbf{x}|^3} \qquad \text{where} \quad S_{ij}(t) = \int T_{ij}(\mathbf{y}, t) \mathrm{d}^3 y$$

and $\dot{}$ denotes differentiation with respect to t. Show further that ρ' scales like $O(m^4)$, where m is the fluctuation Mach number.

(ii) Now consider motion at a single frequency ω. Show that in **two** dimensions ρ' scales like $O(m^{7/2})$.

[In two dimensions the free-space Green's function for Helmholtz equation has the far-field form

$$\frac{\exp(-\mathrm{i} k_0 |\mathbf{x}|)}{\sqrt{k_0 |\mathbf{x}|}}$$

where $k_0 = \omega / c_0$.]

(b) Consider a shock wave (i.e. a surface of discontinuity) in a fluid, with equation $S(\mathbf{x}, t) = 0$, with $S > 0$ on one side of the shock and $S < 0$ on the other side.

(i) By writing the quadrupole distribution in equation (1) in the form

$$T_{ij} = T_{ij}^{+} \mathrm{H}(S) + T_{ij}^{-} \mathrm{H}(-S) \ ,$$

where $T_{ij}^{\pm}$ are continuously differentiable functions and H() is the Heaviside step function, show that the quadrupole terms present separately on either side of the shock are augmented by extra sources located on the shock.

(ii) Now consider one-dimensional flow in which a shock, located at $x = Vt$ with V constant, separates regions of uniform flow (fluid density, pressure and speed ρ_1, p_1, u_1 and ρ_2, p_2, u_2 in $x > Vt$ and $x < Vt$ respectively). Show that Lighthill's equation is now

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x^2} = Q \delta'(x - Vt) \ ,$$

where the quantity Q is to be determined.

2 (a) Describe the use of the Wiener-Hopf technique to solve the Sommerfeld problem of diffraction of a plane wave by an edge, i.e. solve

$$(\nabla^2 + k_0^2)\phi = 0$$

subject to

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi_i}{\partial y} = 0 \quad \text{on} \quad y = 0, x < 0 \ ,$$

where

$$\phi_i = \exp(+\mathrm{i}k_0 x \cos\theta_0 + \mathrm{i}k_0 y \sin\theta_0 + \mathrm{i}\omega t)$$

is the incident potential, $\phi(x, y)\exp(\mathrm{i}\omega t)$ is the scattered potential and $k_0 = \omega/c_0$.

Your answer should include a derivation of the geometrical optics field, and a demonstration that the far-field form of the diffracted field is

$$\left(\frac{2}{\pi k_0 r}\right)^{1/2} \frac{\sin(\theta_0/2)\sin(\theta/2)}{\cos\theta + \cos\theta_0} \exp(-\mathrm{i}k_0 r - \mathrm{i}\pi/4) \ .$$

You need not consider the Fresnel regions around the geometrical optics boundaries. You may quote without proof the result

$$\int_\Gamma f(k)\exp(\mathrm{i}kr\cos\theta - \gamma r|\sin\theta|)\mathrm{d}k \sim \left(\frac{2k_0\pi}{r}\right)^{1/2} f(k_0\cos\theta)|\sin\theta|\exp(-\mathrm{i}k_0 r + \mathrm{i}\pi/4)$$

as $r \to \infty$, where $\gamma = \sqrt{k^2 - k_0^2}$, and Γ is the steepest descent contour (which crosses the real k axis at $k = k_0\cos\theta$ and $k = k_0\sec\theta$).

(b) Consider the semi-infinite duct formed by the two rigid plates $y = \pm h$, $x < 0$. A duct mode with potential

$$\phi_i = \cos(n\pi y/h)\exp(\mathrm{i}\omega t - \mathrm{i}kx) \ ,$$

where $k = \sqrt{k_0^2 - n^2\pi^2/h^2}$ is diffracted by the two edges.

(i) By considering the duct mode to be a superposition of two plane waves propagating in positive and negative y directions, and using the answer to part (a) above, find the diffracted far field to leading order in large $k_0 h$ as a sum of the diffracted fields from each edge.

(ii) Consider the directions

$$\theta = \pm\tan^{-1}(n\pi/kh) \ ,$$

where θ is the observer angle relative to the positive x axis. Comment on the value of the diffracted field found in (i) as compared to the value of the diffracted field from a single edge.

3 (a) Consider the differential equation

$$\frac{d^2x}{dt^2} + (1 + f\epsilon^2 + \epsilon \cos t)x = 0 \ ,$$

subject to $x = 1$, $dx/dt = 0$ at $t = 0$, where $\epsilon \ll 1$ and f is an $O(1)$ constant. Show how the method of multiple scales can be used to find the leading-order approximation to $x(t;\epsilon)$ which is uniformly valid for $t \leq O(1/\epsilon^2)$. For what range of values of the parameter f is your solution stable? Write down the leading-order solution explicitly in this case.

(b) A slowly-varying duct in two dimensions lies parallel to the x axis, and has rigid walls given by $y = \pm R(\epsilon x)$, where $\epsilon \ll 1$. The mean sound speed also varies slowly along the duct, with the effect that the acoustic pressure $p \exp(\mathrm{i}\omega t)$ satisfies

$$\nabla . \left(\frac{1}{k_0^2} \nabla p \right) + p = 0 \ ,$$

where $k_0 = k_0(\epsilon x)$. The wall-normal component of ∇p vanishes on the walls. All quantities are nondimensional.

(i) Show that for a propagating duct mode the leading-order approximation for p takes the form

$$A(X) \exp(-\mathrm{i}kx) \cos(n\pi y/R) \quad , \quad k = \sqrt{k_0^2 - (n\pi/R)^2} \ ,$$

where n is an integer and $X = \epsilon x$.

(ii) Find an explicit expression for the amplitude $A(X)$.

4 (a) Consider a shear flow with nondimensional mean velocity $\mathbf{U} = (U(\epsilon y), 0, 0)$ and with uniform nondimensional mean density ρ_0 and sound speed c_0. The equations describing the propagation of sound waves with fluctuation density ρ' and velocity $\mathbf{u}'$ are

$$\frac{\partial \rho'}{\partial t} + \mathbf{U}.\nabla\rho' + \rho_0 \nabla.\mathbf{u}' = 0 \ ,$$

and

$$\rho_0 \left(\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{U}.\nabla\mathbf{u}' + \mathbf{u}'.\nabla\mathbf{U} \right) = -\nabla(c_0^2 \rho') \ .$$

By writing the fluctuating quantities in the form

$$A(\mathbf{X}) \exp(\mathrm{i}\omega t - \mathrm{i}\theta(\mathbf{X})/\epsilon) \ ,$$

show that

$$(\omega - \mathbf{U}.\nabla\theta)^2 = c_0^2 (\nabla\theta)^2 \ . \qquad (*)$$

High-frequency sound is generated on a flat rigid surface underneath a free stream. Explain briefly, by means of a sketch, what implications $(*)$ has for the direction of propagation through the wall boundary layer.

(b) You are given that Burgers' equation

$$q_Z - q q_\theta = \delta q_{\theta\theta}$$

is related to the diffusion equation

$$\psi_Z = \delta \psi_{\theta\theta}$$

by the Cole-Hopf transformation

$$q = 2\delta \frac{\psi_\theta}{\psi} \ ,$$

and that the general solution of the diffusion equation is

$$\psi(\theta, Z) = \frac{1}{\sqrt{4\pi\delta Z}} \int_{-\infty}^{\infty} \psi(\theta', 0) \exp(-(\theta - \theta')^2 / 4\delta Z) \mathrm{d}\theta' \ .$$

(i) For the initial data $q(\theta, Z = 0) = \sin\theta$, show that in the limit $\delta \ll 1$

$$q(\theta, Z) \sim 4\delta \sin\theta \exp(-\delta Z) \qquad \text{when} \quad \delta Z \gg 1 \ .$$

[You may use the identity

$$\exp(x \cos\theta) = \sum_{n=0}^{\infty} \epsilon_n \mathrm{I}_n(x) \cos n\theta$$

where $\epsilon_0 = 1$, $\epsilon_n = 2, n \geq 1$, and $\mathrm{I}_n(x)$ is the modified Bessel function such that $\mathrm{I}_n(-x) = (-1)^n \mathrm{I}_n(x)$ and $\mathrm{I}_n(x) \sim \exp(x)/(2\pi x)^{\frac{1}{2}}$ as $x \to \infty$.]

(ii) The Fay solution of Burgers' equation is

$$q(\theta, Z) = 2\delta \sum_{n=1}^{\infty} \frac{\sin n\theta}{\sinh(n\delta Z)} \ .$$

Show that the Fay solution has the same behaviour for $\delta Z \gg 1$ as found in (i). Show further that it satisfies

$$q(\theta, Z) \sim \frac{\pi}{Z} \tanh(\pi\theta/2\delta Z)$$

when $\theta \ll 1$, $\delta Z \ll 1$.

[Hint: You may find it helpful to re-express $1/\sinh(n\delta Z)$ as a geometrical series in odd powers of $\exp(-\delta Z)$ and then swap the orders of the summation. You are given the identity

$$\tanh(\pi x/2) \equiv \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2} \ .$$
]

END OF PAPER