UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 10 June, 2011 9:00 am to 12:00 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, ..., ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

(i) Prove that there are infinitely many primes.

(ii) Prove that arbitrarily large gaps can occur between consecutive primes.

2F Topics in Analysis

(a) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map such that $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ of γ about the origin. State precisely a theorem about homotopy invariance of the winding number.

(b) Let $B = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and let $f : B \to \mathbb{C}$ be a continuous map satisfying

$$|f(z) - z| \leqslant 1$$

for each $z \in \partial B$.

(i) For $0 \leqslant t \leqslant 1$, let $\gamma(t) = f(e^{2\pi i t})$. If $\gamma(t) \neq 0$ for each $t \in [0,1]$, prove that $w(\gamma; 0) = 1$.
[*Hint: Consider a suitable homotopy between* γ *and the map* $\gamma_1(t) = e^{2\pi i t}$, $0 \leqslant t \leqslant 1$.]

(ii) Deduce that $f(z) = 0$ for some $z \in B$.

3G Geometry and Groups

Define inversion in a circle Γ on the Riemann sphere. You should show from your definition that inversion in Γ exists and is unique.

Prove that the composition of an even number of inversions is a Möbius transformation of the Riemann sphere and that every Möbius transformation is the composition of an even number of inversions.

4G Coding and Cryptography

Describe a scheme for sending messages based on quantum theory which is not vulnerable to eavesdropping. You may ignore engineering problems.

5J Statistical Modelling

The numbers of ear infections observed among beach and non-beach (mostly pool) swimmers were recorded, along with explanatory variables: frequency, location, age, and sex. The data are aggregated by group, with a total of 24 groups defined by the explanatory variables.

`freq`	F = frequent, NF = infrequent
`loc`	NB = non-beach, B = beach
`age`	15-19, 20-24, 24-29
`sex`	F = female, M = male
`count`	the number of infections reported over a fixed time period
`n`	the total number of swimmers

The data look like this:

```
   count  n freq loc sex   age
1     68 31    F  NB   M 15-19
2     14  4    F  NB   F 15-19
3     35 12    F  NB   M 20-24
4     16 11    F  NB   F 20-24
[...]
23     5 15   NF   B   M 25-29
24     6  6   NF   B   F 25-29
```

Let μ_j denote the expected number of ear infections of a person in group j. Explain why it is reasonable to model $\texttt{count}_j$ as Poisson with mean $n_j\mu_j$.

We fit the following Poisson model:

$$\log(\mathbb{E}(\texttt{count}_j)) = \log(n_j\mu_j) = \log(n_j) + \mathbf{x}_j\beta,$$

where $\log(n_j)$ is an offset, i.e. an explanatory variable with known coefficient 1.

R produces the following (abbreviated) summary for the main effects model:

```
Call:
glm(formula = count ~ freq + loc + age + sex, family = poisson, offset = log(n))
[...]
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  0.48887    0.12271   3.984 6.78e-05 ***
freqNF      -0.61149    0.10500  -5.823 5.76e-09 ***
locNB        0.53454    0.10668   5.011 5.43e-07 ***
age20-24    -0.37442    0.12836  -2.917  0.00354 **
age25-29    -0.18973    0.13009  -1.458  0.14473
sexM        -0.08985    0.11231  -0.800  0.42371
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
[...]
```

Why are expressions freqF, locB, age15-19, and sexF not listed?

Suppose that we plan to observe a group of 20 female, non-frequent, beach swimmers, aged 20-24. Give an expression (using the coefficient estimates from the model fitted above) for the expected number of ear infections in this group.

Now, suppose that we allow for interaction between variables `age` and `sex`. Give the R command for fitting this model. We test for the effect of this interaction by producing the following (abbreviated) ANOVA table:

```
  Resid. Df Resid. Dev Df Deviance P(>|Chi|)
1        18     51.714
2        16     44.319  2   7.3948   0.02479 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Briefly explain what test is performed, and what you would conclude from it. Does either of these models fit the data well?

6B Mathematical Biology

A neglected flower garden contains M_n marigolds in the summer of year n. On average each marigold produces γ seeds through the summer. Seeds may germinate after one or two winters. After three winters or more they will not germinate. Each winter a fraction $1-\alpha$ of all seeds in the garden are eaten by birds (with no preference to the age of the seed). In spring a fraction μ of seeds that have survived one winter and a fraction ν of seeds that have survived two winters germinate. Finite resources of water mean that the number of marigolds growing to maturity from S germinating seeds is $\mathcal{N}(S)$, where $\mathcal{N}(S)$ is an increasing function such that $\mathcal{N}(0) = 0$, $\mathcal{N}'(0) = 1$, $\mathcal{N}'(S)$ is a decreasing function of S and $\mathcal{N}(S) \to N_{max}$ as $S \to \infty$.

Show that M_n satisfies the equation

$$M_{n+1} = \mathcal{N}(\alpha\mu\gamma M_n + \nu\gamma\alpha^2(1-\mu)M_{n-1}) .$$

Write down an equation for the number M_* of marigolds in a steady state. Show graphically that there are two solutions, one with $M_* = 0$ and the other with $M_* > 0$ if

$$\alpha\mu\gamma + \nu\gamma\alpha^2(1-\mu) > 1 .$$

Show that the $M_* = 0$ steady-state solution is unstable to small perturbations in this case.

7C Dynamical Systems

(i) Explain the use of the energy balance method for describing approximately the behaviour of nearly Hamiltonian systems.

(ii) Consider the nearly Hamiltonian dynamical system

$$\ddot{x} + \epsilon\dot{x}(-1 + \alpha x^2 - \beta x^4) + x = 0\,, \qquad 0 < \epsilon \ll 1\,,$$

where α and β are positive constants. Show that, for sufficiently small ϵ, the system has periodic orbits if $\alpha^2 > 8\beta$, and no periodic orbits if $\alpha^2 < 8\beta$. Show that in the first case there are two periodic orbits, and determine their approximate size and their stability.

What can you say about the existence of periodic orbits when $\alpha^2 = 8\beta$?

[You may assume that

$$\int_0^{2\pi} \sin^2 t\, dt = \pi\,, \quad \int_0^{2\pi} \sin^2 t \cos^2 t\, dt = \frac{\pi}{4}\,, \quad \int_0^{2\pi} \sin^2 t \cos^4 t\, dt = \frac{\pi}{8}\,.$$]

8E Further Complex Methods

Let $F(z)$ be defined by

$$F(z) = \int_0^{\infty} \frac{e^{-zt}}{1+t^2}\, dt, \quad |\arg z| < \frac{\pi}{2}\,.$$

Let $\tilde{F}(z)$ be defined by

$$\tilde{F}(z) = \mathcal{P}\int_0^{\infty e^{-\frac{i\pi}{2}}} \frac{e^{-z\zeta}}{1+\zeta^2} d\zeta\,, \quad 0 < \arg z < \pi\,,$$

where $\mathcal{P}$ denotes principal value integral and the contour is the negative imaginary axis.

By computing $F(z) - \tilde{F}(z)$, obtain a formula for the analytic continuation of $F(z)$ for $\frac{\pi}{2} \leqslant \arg z < \pi$.

9C Classical Dynamics

(i) A dynamical system is described by the Hamiltonian $H(q_i, p_i)$. Define the Poisson bracket $\{f, g\}$ of two functions $f(q_i, p_i, t)$, $g(q_i, p_i, t)$. Assuming the Hamiltonian equations of motion, find an expression for df/dt in terms of the Poisson bracket.

(ii) A one-dimensional system has the Hamiltonian

$$H = p^2 + \frac{1}{q^2}\,.$$

Show that $u = pq - 2Ht$ is a constant of the motion. Deduce the form of $(q(t), p(t))$ along a classical path, in terms of the constants u and H.

10E Cosmology

The equilibrium number density of fermions at temperature T is

$$n = \frac{4\pi g_s}{h^3} \int_0^\infty \frac{p^2 dp}{\exp[(\epsilon(p) - \mu)/kT] + 1}\,,$$

where g_s is the spin degeneracy and $\epsilon(p) = c\sqrt{p^2 + m^2c^2}$. For a non-relativistic gas with $pc \ll mc^2$ and $kT \ll mc^2 - \mu$, show that the number density becomes

$$n = g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp[(\mu - mc^2)/kT]\,. \qquad (*)$$

[You may assume that $\int_0^\infty dx\, x^2 e^{-x^2/\alpha} = (\sqrt{\pi}/4)\, \alpha^{3/2}$ for $\alpha > 0$.]

Before recombination, equilibrium is maintained between neutral hydrogen, free electrons, protons and photons through the interaction

$$p + e^- \leftrightarrow H + \gamma\,.$$

Using the non-relativistic number density $(*)$, deduce Saha's equation relating the electron and hydrogen number densities,

$$\frac{n_e^2}{n_H} \approx \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp(-I/kT)\,,$$

where $I = (m_p + m_e - m_H)c^2$ is the ionization energy of hydrogen. State clearly any assumptions you have made.

SECTION II

11I Number Theory

(i) Prove the law of reciprocity for the Jacobi symbol. You may assume the law of reciprocity for the Legendre symbol.

(ii) Let n be an odd positive integer which is not a square. Prove that there exists an odd prime p with $\left(\frac{n}{p}\right) = -1$.

12G Geometry and Groups

Define a *lattice* in $\mathbb{R}^2$ and the *rank* of such a lattice.

Let Λ be a rank 2 lattice in $\mathbb{R}^2$. Choose a vector $\boldsymbol{w}_1 \in \Lambda \setminus \{\mathbf{0}\}$ with $||\boldsymbol{w}_1||$ as small as possible. Then choose $\boldsymbol{w}_2 \in \Lambda \setminus \mathbb{Z}\boldsymbol{w}_1$ with $||\boldsymbol{w}_2||$ as small as possible. Show that $\Lambda = \mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$.

Suppose that $\boldsymbol{w}_1$ is the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Draw the region of possible values for $\boldsymbol{w}_2$.

Suppose that Λ also equals $\mathbb{Z}\boldsymbol{v}_1 + \mathbb{Z}\boldsymbol{v}_2$. Prove that

$$\boldsymbol{v}_1 = a\boldsymbol{w}_1 + b\boldsymbol{w}_2 \qquad \text{and} \qquad \boldsymbol{v}_2 = c\boldsymbol{w}_1 + d\boldsymbol{w}_2\,,$$

for some integers a, b, c, d with $ad - bc = \pm 1$.

13J Statistical Modelling

Consider the general linear model $Y = X\beta + \epsilon$, where the $n \times p$ matrix X has full rank $p \leqslant n$, and where ϵ has a multivariate normal distribution with mean zero and covariance matrix $\sigma^2 I_n$. Write down the likelihood function for β, σ^2 and derive the maximum likelihood estimators $\hat{\beta}, \hat{\sigma}^2$ of β, σ^2. Find the distribution of $\hat{\beta}$. Show further that $\hat{\beta}$ and $\hat{\sigma}^2$ are independent.

14C Dynamical Systems

(i) State and prove Lyapunov's First Theorem, and state (without proof) La Salle's Invariance Principle. Show by example how the latter result can be used to prove asymptotic stability of a fixed point even when a strict Lyapunov function does not exist.

(ii) Consider the system

$$\dot{x} = -x + 2y + x^3 + 2x^2y + 2xy^2 + 2y^3 \, ,$$
$$\dot{y} = -y - x - 2x^3 + \frac{1}{2}x^2y - 3xy^2 + y^3 \, .$$

Show that the origin is asymptotically stable and that the basin of attraction of the origin includes the region $x^2 + 2y^2 < 2/3$.

15C Classical Dynamics

Given a Hamiltonian system with variables (q_i, p_i), $i = 1, \ldots, n$, state the definition of a canonical transformation

$$(q_i, p_i) \to (Q_i, P_i)\,,$$

where $\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{p}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p}, t)$. Write down a matrix equation that is equivalent to the condition that the transformation is canonical.

Consider a harmonic oscillator of unit mass, with Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)\,.$$

Write down the Hamilton–Jacobi equation for Hamilton's principal function $S(q, E, t)$, and deduce the Hamilton–Jacobi equation

$$\frac{1}{2}\left[\left(\frac{\partial W}{\partial q}\right)^2 + \omega^2 q^2\right] = E \qquad (1)$$

for Hamilton's characteristic function $W(q, E)$.

Solve (1) to obtain an integral expression for W, and deduce that, at energy E,

$$S = \sqrt{2E}\int dq\sqrt{\left(1 - \frac{\omega^2 q^2}{2E}\right)} - Et\,. \qquad (2)$$

Let $\alpha = E$, and define the angular coordinate

$$\beta = \left(\frac{\partial S}{\partial E}\right)_{q,t}\,.$$

You may assume that (2) implies

$$t + \beta = \left(\frac{1}{\omega}\right)\arcsin\left(\frac{\omega q}{\sqrt{2E}}\right)\,.$$

Deduce that

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{(2E - \omega^2 q^2)}\,,$$

from which

$$p = \sqrt{2E}\cos[\omega(t + \beta)]\,.$$

Hence, or otherwise, show that the transformation from variables (q, p) to (α, β) is canonical.

16H Logic and Set Theory

Define the sets V_α for ordinals α. Show that each V_α is transitive. Show also that $V_\alpha \subseteq V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

For which ordinals α does there exist a set x such that the power-set of x has rank α? [You may assume standard properties of rank.]

17F Graph Theory

(i) Given a positive integer k, show that there exists a positive integer n such that, whenever the edges of the complete graph K_n are coloured with k colours, there exists a monochromatic triangle.

Denote the least such n by $f(k)$. Show that $f(k) \leqslant 3 \cdot k!$ for all k.

(ii) You may now assume that $f(2) = 6$ and $f(3) = 17$.

Let H denote the graph of order 4 consisting of a triangle together with one extra edge. Given a positive integer k, let $g(k)$ denote the least positive integer n such that, whenever the edges of the complete graph K_n are coloured with k colours, there exists a monochromatic copy of H. By considering the edges from one vertex of a monochromatic triangle in K_7, or otherwise, show that $g(2) \leqslant 7$. By exhibiting a blue-yellow colouring of the edges of K_6 with no monochromatic copy of H, show that in fact $g(2) = 7$.

What is $g(3)$? Justify your answer.

18H Galois Theory

Let K be a field of characteristic 0, and let $P(X) = X^4 + bX^2 + cX + d$ be an *irreducible* quartic polynomial over K. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be its roots in an algebraic closure of K, and consider the Galois group $\mathrm{Gal}(P)$ (the group $\mathrm{Gal}(F/K)$ for a splitting field F of P over K) as a subgroup of S_4 (the group of permutations of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$).

Suppose that $\mathrm{Gal}(P)$ contains $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$.

(i) List all possible $\mathrm{Gal}(P)$ up to isomorphism. [*Hint: there are 4 cases, with orders 4, 8, 12 and 24.*]

(ii) Let $Q(X)$ be the *resolvent cubic* of P, i.e. a cubic in $K[X]$ whose roots are $-(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)$, $-(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)$ and $-(\alpha_1+\alpha_4)(\alpha_2+\alpha_3)$. Construct a natural surjection $\mathrm{Gal}(P) \to \mathrm{Gal}(Q)$, and find $\mathrm{Gal}(Q)$ in each of the four cases found in (i).

(iii) Let $\Delta \in K$ be the discriminant of Q. Give a criterion to determine $\mathrm{Gal}(P)$ in terms of Δ and the factorisation of Q in $K[X]$.

(iv) Give a specific example of P where $\mathrm{Gal}(P)$ is abelian.

19I Representation Theory

Define the groups SU(2) and SO(3).

Show that $G = \mathrm{SU}(2)$ acts on the vector space of 2×2 complex matrices of the form

$$V = \left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Denote the corresponding representation of SU(2) on V by ρ.

Prove the following assertions about this action:

(i) The subspace V is isomorphic to $\mathbb{R}^3$.

(ii) The pairing $(A, B) \mapsto -\mathrm{tr}(AB)$ defines a positive definite non-degenerate SU(2)-invariant bilinear form.

(iii) The representation ρ maps G into SO(3). [You may assume that for any compact group H, and any $n \in \mathbb{N}$, there is a continuous group homomorphism $H \to \mathrm{O}(n)$ if and only if H has an n-dimensional representation over $\mathbb{R}$.]

Write down an orthonormal basis for V and use it to show that ρ is surjective with kernel $\{\pm I\}$.

Use the isomorphism $\mathrm{SO}(3) \cong G/\{\pm I\}$ to write down a list of irreducible representations of SO(3) in terms of irreducibles for SU(2). [Detailed explanations are not required.]

20F Number Fields

(i) Prove that the ring of integers $\mathcal{O}_K$ in a real quadratic field K contains a non-trivial unit. Any general results about lattices and convex bodies may be assumed.

(ii) State the general version of Dirichlet's unit theorem.

(iii) Show that for $K = \mathbb{Q}(\sqrt{7})$, $8 + 3\sqrt{7}$ is a fundamental unit in $\mathcal{O}_K$.

[You may not use results about continued fractions unless you prove them.]

21H Algebraic Topology

State the Mayer–Vietoris theorem, and use it to calculate, for each integer $q > 1$, the homology group of the space X_q obtained from the unit disc $B^2 \subseteq \mathbb{C}$ by identifying pairs of points (z_1, z_2) on its boundary whenever $z_1^q = z_2^q$. [You should construct an explicit triangulation of X_q.]

Show also how the theorem may be used to calculate the homology groups of the suspension SK of a connected simplicial complex K in terms of the homology groups of K, and of the wedge union $X \vee Y$ of two connected polyhedra. Hence show that, for any finite sequence $(G_1, G_2, \ldots, G_n)$ of finitely-generated abelian groups, there exists a polyhedron X such that $H_0(X) \cong \mathbb{Z}$, $H_i(X) \cong G_i$ for $1 \leqslant i \leqslant n$ and $H_i(X) = 0$ for $i > n$. [You may assume the structure theorem which asserts that any finitely-generated abelian group is isomorphic to a finite direct sum of (finite or infinite) cyclic groups.]

22G Linear Analysis

State Urysohn's Lemma. State and prove the Tietze Extension Theorem.

Let X, Y be two topological spaces. We say that the extension property holds if, whenever $S \subseteq X$ is a closed subset and $f : S \to Y$ is a continuous map, there is a continuous function $\tilde{f} : X \to Y$ with $\tilde{f}|_S = f$.

For each of the following three statements, say whether it is true or false. Briefly justify your answers.

1. If X is a metric space and $Y = [-1, 1]$ then the extension property holds.
2. If X is a compact Hausdorff space and $Y = \mathbb{R}$ then the extension property holds.
3. If X is an arbitrary topological space and $Y = [-1, 1]$ then the extension property holds.

23H Algebraic Geometry

Let X be a smooth projective curve over an algebraically closed field k.

State the Riemann–Roch theorem, briefly defining all the terms that appear.

Now suppose X has genus 1, and let $P_\infty \in X$.

Compute $\mathcal{L}(nP_\infty)$ for $n \leqslant 6$. Show that ϕ_{3P_∞} defines an isomorphism of X with a smooth plane curve in $\mathbb{P}^2$ which is defined by a polynomial of degree 3.

24I Differential Geometry

Define what is meant by a *geodesic*. Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the *geodesic curvature* k_g of a smooth curve $\gamma : I \to S$ parametrized by arc-length.

Explain without detailed proofs what are the *exponential map* $\exp_p$ and the *geodesic polar coordinates* (r, θ) at $p \in S$. Determine the derivative $d(\exp_p)_0$. Prove that the coefficients of the first fundamental form of S in the geodesic polar coordinates satisfy

$$E = 1\,, \quad F = 0\,, \quad G(0,\theta) = 0\,, \quad (\sqrt{G})_r(0,\theta) = 1\,.$$

State the global Gauss–Bonnet formula for compact surfaces with boundary. [You should identify all terms in the formula.]

Suppose that S is homeomorphic to a cylinder $S^1 \times \mathbb{R}$ and has negative Gaussian curvature at each point. Prove that S has at most one simple (i.e. without self-intersections) closed geodesic.

[Basic properties of geodesics may be assumed, if accurately stated.]

25K Probability and Measure

(i) State and prove Fatou's lemma. State and prove Lebesgue's dominated convergence theorem. [You may assume the monotone convergence theorem.]

In the rest of the question, let f_n be a sequence of integrable functions on some measure space $(E, \mathcal{E}, \mu)$, and assume that $f_n \to f$ almost everywhere, where f is a given integrable function. We also assume that $\int |f_n| d\mu \to \int |f| d\mu$ as $n \to \infty$.

(ii) Show that $\int f_n^+ d\mu \to \int f^+ d\mu$ and that $\int f_n^- d\mu \to \int f^- d\mu$, where $\phi^+ = \max(\phi, 0)$ and $\phi^- = \max(-\phi, 0)$ denote the positive and negative parts of a function ϕ.

(iii) Here we assume also that $f_n \geqslant 0$. Deduce that $\int |f - f_n| d\mu \to 0$.

26J Applied Probability

At an M/G/1 queue, the arrival times form a Poisson process of rate λ while service times S_1, S_2, ... are independent of each other and of the arrival times and have a common distribution G with mean $\mathbb{E}S_1 < +\infty$.

(i) Show that the random variables Q_n giving the number of customers left in the queue at departure times form a Markov chain.

(ii) Specify the transition probabilities of this chain as integrals in $\mathrm{d}G(t)$ involving parameter λ. [No proofs are needed.]

(iii) Assuming that $\rho = \lambda\mathbb{E}S_1 < 1$ and the chain (Q_n) is positive recurrent, show that its stationary distribution $(\pi_k,\ k \geqslant 0)$ has the generating function given by

$$\sum_{k\geqslant 0} \pi_k s^k = \frac{(1-\rho)(s-1)g(s)}{s-g(s)}\,, \quad |s| \leqslant 1\,,$$

for an appropriate function g, to be specified.

(iv) Deduce that, in equilibrium, Q_n has the mean value

$$\rho + \frac{\lambda^2 \mathbb{E}S_1^2}{2(1-\rho)}\,.$$

27K Principles of Statistics

What does it mean to say that a $(1 \times p)$ random vector ξ has a *multivariate normal distribution*?

Suppose $\xi = (X, Y)$ has the bivariate normal distribution with mean vector $\mu = (\mu_X, \mu_Y)$, and dispersion matrix

$$\Sigma = \begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix}.$$

Show that, with $\beta := \sigma_{XY}/\sigma_{XX}$, $Y - \beta X$ is independent of X, and thus that the conditional distribution of Y given X is normal with mean $\mu_Y + \beta(X - \mu_X)$ and variance $\sigma_{YY \cdot X} := \sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}$.

For $i = 1, \ldots, n$, $\xi_i = (X_i, Y_i)$ are independent and identically distributed with the above distribution, where all elements of μ and Σ are unknown. Let

$$S = \begin{pmatrix} S_{XX} & S_{XY} \\ S_{XY} & S_{YY} \end{pmatrix} := \sum_{i=1}^{n} (\xi_i - \overline{\xi})^{\mathrm{T}} (\xi_i - \overline{\xi}),$$

where $\overline{\xi} := n^{-1} \sum_{i=1}^{n} \xi_i$.

The *sample correlation coefficient* is $r := S_{XY}/\sqrt{S_{XX} S_{YY}}$. Show that the distribution of r depends only on the population correlation coefficient $\rho := \sigma_{XY}/\sqrt{\sigma_{XX}\sigma_{YY}}$.

Student's t-statistic (on $n-2$ degrees of freedom) for testing the null hypothesis $H_0 : \beta = 0$ is

$$t := \frac{\widehat{\beta}}{\sqrt{S_{YY \cdot X}/(n-2) S_{XX}}},$$

where $\widehat{\beta} := S_{XY}/S_{XX}$ and $S_{YY \cdot X} := S_{YY} - S_{XY}^2/S_{XX}$. Its density when H_0 is true is

$$p(t) = C \left(1 + \frac{t^2}{n-2}\right)^{-\frac{1}{2}(n-1)},$$

where C is a constant that need not be specified.

Express t in terms of r, and hence derive the density of r when $\rho = 0$.

How could you use the sample correlation r to test the hypothesis $\rho = 0$?

28K Optimization and Control

Describe the type of optimal control problem that is amenable to analysis using Pontryagin's Maximum Principle.

A firm has the right to extract oil from a well over the interval $[0, T]$. The oil can be sold at price £p per unit. To extract oil at rate u when the remaining quantity of oil in the well is x incurs cost at rate £u^2/x. Thus the problem is one of maximizing

$$\int_0^T \left[pu(t) - \frac{u(t)^2}{x(t)} \right] dt \,,$$

subject to $dx(t)/dt = -u(t)$, $u(t) \geqslant 0$, $x(t) \geqslant 0$. Formulate the Hamiltonian for this problem.

Explain why $\lambda(t)$, the adjoint variable, has a boundary condition $\lambda(T) = 0$.

Use Pontryagin's Maximum Principle to show that under optimal control

$$\lambda(t) = p - \frac{1}{1/p + (T-t)/4}$$

and

$$\frac{dx(t)}{dt} = -\frac{2px(t)}{4 + p(T-t)} \,.$$

Find the oil remaining in the well at time T, as a function of $x(0)$, p, and T,

29J Stochastic Financial Models

In a two-period model, two agents enter a negotiation at time 0. Agent j knows that he will receive a random payment X_j at time 1 ($j = 1, 2$), where the joint distribution of (X_1, X_2) is known to both agents, and $X_1 + X_2 > 0$. At the outcome of the negotiation, there will be an agreed *risk transfer* random variable Y which agent 1 will pay to agent 2 at time 1. The objective of agent 1 is to maximize $EU_1(X_1 - Y)$, and the objective of agent 2 is to maximize $EU_2(X_2 + Y)$, where the functions U_j are strictly increasing, strictly concave, C^2, and have the properties that

$$\lim_{x \downarrow 0} U_j'(x) = +\infty\,, \qquad \lim_{x \uparrow \infty} U_j'(x) = 0\,.$$

Show that, unless there exists some $\lambda \in (0, \infty)$ such that

$$\frac{U_1'(X_1 - Y)}{U_2'(X_2 + Y)} = \lambda \quad \text{almost surely,} \qquad (*)$$

the risk transfer Y could be altered to the benefit of both agents, and so would not be the conclusion of the negotiation.

Show that, for given $\lambda > 0$, the relation $(*)$ determines a unique risk transfer $Y = Y_\lambda$, and that $X_2 + Y_\lambda$ is a function of $X_1 + X_2$.

30A Partial Differential Equations

Consider the functional

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega F(u, x) \, dx\,,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $F : \mathbb{R} \times \Omega \to \mathbb{R}$ is smooth. Assume that $F(u, x)$ is convex in u for all $x \in \Omega$ and that there is a $K > 0$ such that

$$-K \leqslant F(v, x) \leqslant K\left(|v|^2 + 1\right) \qquad \forall v \in \mathbb{R},\, x \in \Omega\,.$$

(i) Prove that E is well-defined on $H_0^1(\Omega)$, bounded from below and strictly convex. Assume without proof that E is weakly lower-semicontinuous. State this property. Conclude the existence of a unique minimizer of E.

(ii) Which elliptic boundary value problem does the minimizer solve?

31A Asymptotic Methods

Determine the range of the integer n for which the equation

$$\frac{d^2y}{dz^2} = z^n y$$

has an essential singularity at $z = \infty$.

Use the Liouville–Green method to find the leading asymptotic approximation to two independent solutions of

$$\frac{d^2y}{dz^2} = z^3 y\,,$$

for large $|z|$. Find the Stokes lines for these approximate solutions. For what range of $\arg z$ is the approximate solution which decays exponentially along the positive z-axis an asymptotic approximation to an exact solution with this exponential decay?

32D Principles of Quantum Mechanics

The quantum-mechanical observable Q has just two orthonormal eigenstates $|1\rangle$ and $|2\rangle$ with eigenvalues -1 and 1, respectively. The operator Q' is defined by $Q' = Q + \epsilon T$, where

$$T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Defining orthonormal eigenstates of Q' to be $|1'\rangle$ and $|2'\rangle$ with eigenvalues q_1', q_2', respectively, consider a perturbation to first order in $\epsilon \in \mathbb{R}$ for the states

$$|1'\rangle = a_1|1\rangle + a_2\epsilon|2\rangle\,, \qquad |2'\rangle = b_1|2\rangle + b_2\epsilon|1\rangle\,,$$

where a_1, a_2, b_1, b_2 are complex coefficients. The real eigenvalues are also expanded to first order in ϵ:

$$q_1' = -1 + c_1\epsilon\,, \qquad q_2' = 1 + c_2\epsilon\,.$$

From first principles, find a_1, a_2, b_1, b_2, c_1, c_2.

Working exactly to all orders, find the real eigenvalues q_1', q_2' directly. Show that the exact eigenvectors of Q' may be taken to be of the form

$$A_j(\epsilon)\begin{pmatrix} 1 \\ -i(1 + Bq_j')/\epsilon \end{pmatrix},$$

finding $A_j(\epsilon)$ and the real numerical coefficient B in the process.

By expanding the exact expressions, again find a_1, a_2, b_1, b_2, c_1, c_2, verifying the perturbation theory results above.

33E Applications of Quantum Mechanics

A particle of charge $-e$ and mass m moves in a magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ and in an electric potential $\phi(\boldsymbol{x},t)$. The time-dependent Schrödinger equation for the particle's wavefunction $\Psi(\boldsymbol{x},t)$ is

$$i\hbar\left(\frac{\partial}{\partial t}-\frac{ie}{\hbar}\phi\right)\Psi=-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}+\frac{ie}{\hbar}\boldsymbol{A}\right)^2\Psi\,,$$

where $\boldsymbol{A}$ is the vector potential with $\boldsymbol{B}=\boldsymbol{\nabla}\wedge\boldsymbol{A}$. Show that this equation is invariant under the gauge transformations

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t) &\rightarrow \boldsymbol{A}(\boldsymbol{x},t)+\boldsymbol{\nabla}f(\boldsymbol{x},t)\,,\\ \phi(\boldsymbol{x},t) &\rightarrow \phi(\boldsymbol{x},t)-\tfrac{\partial}{\partial t}f(\boldsymbol{x},t)\,,\end{aligned}$$

where f is an arbitrary function, together with a suitable transformation for Ψ which should be stated.

Assume now that $\partial\Psi/\partial z=0$, so that the particle motion is only in the x and y directions. Let $\boldsymbol{B}$ be the constant field $\boldsymbol{B}=(0,0,B)$ and let $\phi=0$. In the gauge where $\boldsymbol{A}=(-By,0,0)$ show that the stationary states are given by

$$\Psi_k(\boldsymbol{x},t)=\psi_k(\boldsymbol{x})e^{-iEt/\hbar}\,,$$

with

$$\psi_k(\boldsymbol{x})=e^{ikx}\chi_k(y)\,. \qquad (*)$$

Show that $\chi_k(y)$ is the wavefunction for a simple one-dimensional harmonic oscillator centred at position $y_0=\hbar k/eB$. Deduce that the stationary states lie in infinitely degenerate levels (Landau levels) labelled by the integer $n\geqslant 0$, with energy

$$E_n=(2n+1)\frac{\hbar eB}{2m}\,.$$

A uniform electric field $\mathcal{E}$ is applied in the y-direction so that $\phi=-\mathcal{E}y$. Show that the stationary states are given by $(*)$, where $\chi_k(y)$ is a harmonic oscillator wavefunction centred now at

$$y_0=\frac{1}{eB}\left(\hbar k-m\frac{\mathcal{E}}{B}\right)\,.$$

Show also that the eigen-energies are given by

$$E_{n,k}=(2n+1)\frac{\hbar eB}{2m}+e\mathcal{E}y_0+\frac{m\mathcal{E}^2}{2B^2}\,.$$

Why does this mean that the Landau energy levels are no longer degenerate in two dimensions?

34D Statistical Physics

(i) Define the Gibbs free energy for a gas of N particles with pressure p at a temperature T. Explain why it is necessarily proportional to the number of particles N in the system. Given volume V and chemical potential μ, prove that

$$\left.\frac{\partial \mu}{\partial p}\right|_T = \frac{V}{N}\,.$$

(ii) The van der Waals equation of state is

$$\left(p + \frac{aN^2}{V^2}\right)(V - Nb) = Nk_BT\,.$$

Explain the physical significance of the terms with constants a and b. Sketch the isotherms of the van der Waals equation. Show that the critical point lies at

$$k_BT_c = \frac{8a}{27b}\,, \qquad V_c = 3bN\,, \qquad p_c = \frac{a}{27b^2}\,.$$

(iii) Describe the Maxwell construction to determine the condition for phase equilibrium. Hence sketch the regions of the van der Waals isotherm at $T < T_c$ that correspond to metastable and unstable states. Sketch those regions that correspond to stable liquids and stable gases.

(iv) Show that, as the critical point is approached along the co-existence curve,

$$V_{\text{gas}} - V_{\text{liquid}} \sim (T_c - T)^{1/2}\,.$$

Show that, as the critical point is approached along an isotherm,

$$p - p_c \sim (V - V_c)^3\,.$$

35C Electrodynamics

Suppose that there is a distribution of electric charge given by the charge density $\rho(\mathbf{x})$. Develop the multipole expansion, up to quadrupole terms, for the electrostatic potential ϕ and define the dipole and quadrupole moments of the charge distribution.

A tetrahedron has a vertex at $(1,1,1)$ where there is a point charge of strength $3q$. At each of the other vertices located at $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$ there is a point charge of strength $-q$.

What is the dipole moment of this charge distribution?

What is the quadrupole moment?

36D General Relativity

The metric of the Schwarzschild solution is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \qquad (*)$$

Show that, for an incoming radial light ray, the quantity

$$v = t + r + 2M \log \left| \frac{r}{2M} - 1 \right|$$

is constant.

Express ds^2 in terms of r, v, θ and ϕ. Determine the light-cone structure in these coordinates, and use this to discuss the nature of the apparent singularity at $r = 2M$.

An observer is falling radially inwards in the region $r < 2M$. Assuming that the metric for $r < 2M$ is again given by $(*)$, obtain a bound for $d\tau$, where τ is the proper time of the observer, in terms of dr. Hence, or otherwise, determine the maximum proper time that can elapse between the events at which the observer crosses $r = 2M$ and is torn apart at $r = 0$.

37B Fluid Dynamics II

A viscous fluid flows along a slowly varying thin channel between no-slip surfaces at $y = 0$ and $y = h(x,t)$ under the action of a pressure gradient dp/dx. After explaining the approximations and assumptions of lubrication theory, including a comment on the reduced Reynolds number, derive the expression for the volume flux

$$q = \int_0^h u\, dy = -\frac{h^3}{12\mu} \frac{dp}{dx}\,,$$

as well as the equation

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0\,.$$

In peristaltic pumping, the surface $h(x,t)$ has a periodic form in space which propagates at a constant speed c, i.e. $h(x - ct)$, and no net pressure gradient is applied, i.e. the pressure gradient averaged over a period vanishes. Show that the average flux along the channel is given by

$$\langle q \rangle = c \left(\langle h \rangle - \frac{\langle h^{-2} \rangle}{\langle h^{-3} \rangle} \right)\,,$$

where $\langle \cdot \rangle$ denotes an average over one period.

38B Waves

Show that, in the standard notation for one-dimensional flow of a perfect gas, the Riemann invariants $u \pm 2(c - c_0)/(\gamma - 1)$ are constant on characteristics $C_\pm$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c\,.$$

Such a gas occupies the region $x > X(t)$ in a semi-infinite tube to the right of a piston at $x = X(t)$. At time $t = 0$, the piston and the gas are at rest, $X = 0$, and the gas is uniform with $c = c_0$. For $t > 0$ the piston accelerates smoothly in the positive x-direction. Show that, prior to the formation of a shock, the motion of the gas is given parametrically by

$$u(x,t) = \dot{X}(\tau) \quad \text{on} \quad x = X(\tau) + \left[c_0 + \tfrac{1}{2}(\gamma+1)\dot{X}(\tau)\right](t - \tau)\,,$$

in a region that should be specified.

For the case $X(t) = \frac{2}{3}c_0 t^3/T^2$, where $T > 0$ is a constant, show that a shock first forms in the gas when

$$t = \frac{T}{\gamma+1}(3\gamma+1)^{1/2}\,.$$

39A Numerical Analysis

(i) Consider the Poisson equation

$$\nabla^2 u = f\,, \quad -1 \leqslant x, y \leqslant 1\,,$$

with the periodic boundary conditions

$$u(-1,y) = u(1,y)\,, \quad u_x(-1,y) = u_x(1,y)\,, \quad -1 \leqslant y \leqslant 1\,,$$
$$u(x,-1) = u(x,1)\,, \quad u_y(x,-1) = u_y(x,1)\,, \quad -1 \leqslant x \leqslant 1$$

and the normalization condition

$$\int_{-1}^{1}\int_{-1}^{1} u(x,y)\, dx\, dy = 0\,.$$

Moreover, f is analytic and obeys the periodic boundary conditions $f(-1,y) = f(1,y)$, $f(x,-1) = f(x,1)$, $-1 \leqslant x, y \leqslant 1$.

Derive an explicit expression of the approximation of a solution u by means of a spectral method. Explain the term *convergence with spectral speed* and state its validity for the approximation of u.

(ii) Consider the second-order linear elliptic partial differential equation

$$\nabla \cdot (a \nabla u) = f\,, \quad -1 \leqslant x, y \leqslant 1\,,$$

with the periodic boundary conditions and normalization condition specified in (i). Moreover, a and f are given by

$$a(x,y) = \cos(\pi x) + \cos(\pi y) + 3\,, \quad f(x,y) = \sin(\pi x) + \sin(\pi y)\,.$$

[Note that a is a positive analytic periodic function.]

Construct explicitly the linear algebraic system that arises from the implementation of a spectral method to the above equation.

END OF PAPER