MATHEMATICAL TRIPOS Part II 2010

List of Courses

Paper 1, Section II

24G Algebraic Geometry

(i) Let $X = \{(x, y) \in \mathbb{C}^2 \mid x^2 = y^3\}$. Show that X is birational to $\mathbf{A}^1$, but not isomorphic to it.

(ii) Let X be an affine variety. Define the *dimension* of X in terms of the tangent spaces of X.

(iii) Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial, where k is an algebraically closed field of arbitrary characteristic. Show that $\dim Z(f) = n - 1$.

[You may assume the Nullstellensatz.]

Paper 2, Section II

24G Algebraic Geometry

Let $X = X_{n,m,r}$ be the set of $n \times m$ matrices of rank at most r over a field k. Show that $X_{n,m,r}$ is naturally an affine subvariety of $\mathbf{A}^{nm}$ and that $X_{n,m,r}$ is a Zariski closed subvariety of $X_{n,m,r+1}$.

Show that if $r < \min(n, m)$, then 0 is a singular point of X.

Determine the dimension of $X_{5,2,1}$.

Paper 3, Section II

23G Algebraic Geometry

(i) Let X be a curve, and $p \in X$ be a smooth point on X. Define what a *local parameter* at p is.

Now let $f : X \dashrightarrow Y$ be a rational map to a quasi-projective variety Y. Show that if Y is projective, f extends to a morphism defined at p.

Give an example where this fails if Y is not projective, and an example of a morphism $f : \mathbb{C}^2 \setminus \{0\} \to \mathbf{P}^1$ which does not extend to 0.

(ii) Let $V = Z(X_0^8 + X_1^8 + X_2^8)$ and $W = Z(X_0^4 + X_1^4 + X_2^4)$ be curves in $\mathbf{P}^2$ over a field of characteristic not equal to 2. Let $\phi : V \to W$ be the map $[X_0 : X_1 : X_2] \mapsto [X_0^2 : X_1^2 : X_2^2]$. Determine the degree of ϕ, and the ramification e_p for all $p \in V$.

Paper 4, Section II

23G Algebraic Geometry

Let $E \subseteq \mathbf{P}^2$ be the projective curve obtained from the affine curve $y^2 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, where the λ_i are distinct and $\lambda_1 \lambda_2 \lambda_3 \neq 0$.

(i) Show there is a unique point at infinity, P_∞.

(ii) Compute $\operatorname{div}(x)$, $\operatorname{div}(y)$.

(iii) Show $\mathcal{L}(P_\infty) = k$.

(iv) Compute $l(nP_\infty)$ for all n.

[You may *not* use the Riemann–Roch theorem.]

Paper 1, Section II

21H Algebraic Topology

State the path lifting and homotopy lifting lemmas for covering maps. Suppose that X is path connected and locally path connected, that $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are covering maps, and that Y_1 and Y_2 are simply connected. Using the lemmas you have stated, but without assuming the correspondence between covering spaces and subgroups of π_1, prove that Y_1 is homeomorphic to Y_2.

Paper 2, Section II

21H Algebraic Topology

Let G be the finitely presented group $G = \langle a, b \,|\, a^2\, b^3\, a^3\, b^2 = 1 \rangle$. Construct a path connected space X with $\pi_1(X, x) \cong G$. Show that X has a unique connected double cover $\pi : Y \to X$, and give a presentation for $\pi_1(Y, y)$.

Paper 3, Section II

20H Algebraic Topology

Suppose X is a finite simplicial complex and that $H_*(X)$ is a free abelian group for each value of $*$. Using the Mayer-Vietoris sequence or otherwise, compute $H_*(S^1 \times X)$ in terms of $H_*(X)$. Use your result to compute $H_*(T^n)$.

[Note that $T^n = S^1 \times \ldots \times S^1$, where there are n factors in the product.]

Paper 4, Section II

21H Algebraic Topology

State the Snake Lemma. Explain how to define the boundary map which appears in it, and check that it is well-defined. Derive the Mayer-Vietoris sequence from the Snake Lemma.

Given a chain complex C, let $A \subset C$ be the span of all elements in C with grading greater than or equal to n, and let $B \subset C$ be the span of all elements in C with grading less than n. Give a short exact sequence of chain complexes relating A, B, and C. What is the boundary map in the corresponding long exact sequence?

Paper 1, Section II

34B Applications of Quantum Mechanics

Give an account of the variational principle for establishing an upper bound on the ground-state energy, E_0, of a particle moving in a potential $V(x)$ in one dimension.

Explain how an upper bound on the energy of the first excited state can be found in the case that $V(x)$ is a symmetric function.

A particle of mass $2m = \hbar^2$ moves in the potential

$$V(x) = -V_0 \, e^{-x^2}, \qquad V_0 > 0.$$

Use the trial wavefunction

$$\psi(x) = e^{-\frac{1}{2} a x^2},$$

where a is a positive real parameter, to establish the upper bound $E_0 \leqslant E(a)$ for the energy of the ground state, where

$$E(a) = \frac{1}{2} a - V_0 \frac{\sqrt{a}}{\sqrt{1+a}}.$$

Show that, for $a > 0$, $E(a)$ has one zero and find its position.

Show that the turning points of $E(a)$ are given by

$$(1+a)^3 = \frac{V_0^2}{a},$$

and deduce that there is one turning point in $a > 0$ for all $V_0 > 0$.

Sketch $E(a)$ for $a > 0$ and hence deduce that $V(x)$ has at least one bound state for all $V_0 > 0$.

For $0 < V_0 \ll 1$ show that

$$-V_0 < E_0 \leqslant \epsilon(V_0),$$

where $\epsilon(V_0) = -\frac{1}{2} V_0{}^2 + \mathrm{O}(V_0{}^4)$.

[You may use the result that $\int_{-\infty}^{\infty} e^{-bx^2}\, dx = \sqrt{\frac{\pi}{b}}$ for $b > 0$.]

Paper 2, Section II

34B Applications of Quantum Mechanics

A beam of particles of mass m and momentum $p = \hbar k$ is incident along the z-axis. Write down the asymptotic form of the wave function which describes scattering under the influence of a spherically symmetric potential $V(r)$ and which defines the scattering amplitude $f(\theta)$.

Given that, for large r,

$$e^{ikr\cos\theta} \sim \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1) \left(e^{ikr} - (-1)^l e^{-ikr} \right) P_l(\cos\theta)\,,$$

show how to derive the partial-wave expansion of the scattering amplitude in the form

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta)\,.$$

Obtain an expression for the total cross-section, σ.

Let $V(r)$ have the form

$$V(r) = \begin{cases} -V_0\,, & r < a\,, \\ 0\,, & r > a\,, \end{cases}$$

where $V_0 = \dfrac{\hbar^2}{2m}\,\gamma^2$.

Show that the $l = 0$ phase-shift δ_0 satisfies

$$\frac{\tan(ka + \delta_0)}{ka} = \frac{\tan\kappa a}{\kappa a}\,,$$

where $\kappa^2 = k^2 + \gamma^2$.

Assume γ to be large compared with k so that κ may be approximated by γ. Show, using graphical methods or otherwise, that there are values for k for which $\delta_0(k) = n\pi$ for some integer n, which should not be calculated. Show that the smallest value, k_0, of k for which this condition holds certainly satisfies $k_0 < 3\pi/2a$.

Paper 3, Section II

34B Applications of Quantum Mechanics

State Bloch's theorem for a one dimensional lattice which is invariant under translations by a.

A simple model of a crystal consists of a one-dimensional linear array of identical sites with separation a. At the nth site the Hamiltonian, neglecting all other sites, is H_n and an electron may occupy either of two states, $\phi_n(x)$ and $\chi_n(x)$, where

$$H_n \, \phi_n(x) \;=\; E_0 \, \phi_n(x) \,, \qquad\qquad H_n \, \chi_n(x) \;=\; E_1 \, \chi_n(x) \,,$$

and ϕ_n and χ_n are orthonormal. How are $\phi_n(x)$ and $\chi_n(x)$ related to $\phi_0(x)$ and $\chi_0(x)$?

The full Hamiltonian is H and is invariant under translations by a. Write trial wavefunctions $\psi(x)$ for the eigenstates of this model appropriate to a tight binding approximation if the electron has probability amplitudes b_n and c_n to be in the states ϕ_n and χ_n respectively.

Assume that the only non-zero matrix elements in this model are, for all n,

$$(\phi_n, H_n \, \phi_n) \;=\; E_0 \,, \qquad\qquad (\chi_n, H_n \, \chi_n) \;=\; E_1 \,,$$

$$(\phi_n, V \phi_{n\pm1}) \;=\; (\chi_n, V \, \chi_{n\pm1}) \;=\; (\phi_n, V \, \chi_{n\pm1}) \;=\; (\chi_n, V \phi_{n\pm1}) \;=\; -A \,,$$

where $H = H_n + V$ and $A > 0$. Show that the time-dependent Schrödinger equation governing the amplitudes becomes

$$i\hbar \, \dot{b}_n \;=\; E_0 \, b_n - A(b_{n+1} + b_{n-1} + c_{n+1} + c_{n-1}) \,,$$

$$i\hbar \, \dot{c}_n \;=\; E_1 \, c_n - A(c_{n+1} + c_{n-1} + b_{n+1} + b_{n-1}) \,.$$

By examining solutions of the form

$$\begin{pmatrix} b_n \\ c_n \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} e^{\,i(kna - Et/\hbar)} \,,$$

show that the allowed energies of the electron are two bands given by

$$E \;=\; \frac{1}{2}\,(E_0 + E_1 - 4\,A\cos ka) \pm \frac{1}{2}\,\sqrt{(E_0 - E_1)^2 + 16\,A^2\cos^2 ka} \,.$$

Define the *Brillouin zone* for this system and find the energies at the top and bottom of both bands. Hence, show that the energy gap between the bands is

$$\Delta E \;=\; -4\,A + \sqrt{(E_1 - E_0)^2 + 16\,A^2} \,.$$

Show that the wavefunctions $\psi(x)$ satisfy Bloch's theorem.

Describe briefly what are the crucial differences between insulators, conductors and semiconductors.

Paper 4, Section II

33B Applications of Quantum Mechanics

The scattering amplitude for electrons of momentum $\hbar\mathbf{k}$ incident on an atom located at the origin is $f(\hat{\mathbf{r}})$ where $\hat{\mathbf{r}} = \mathbf{r}/r$. Explain why, if the atom is displaced by a position vector $\mathbf{a}$, the asymptotic form of the scattering wave function becomes

$$\psi_{\mathbf{k}}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} + e^{i\mathbf{k}\cdot\mathbf{a}} \frac{e^{ikr'}}{r'} f(\hat{\mathbf{r}}') \sim e^{i\mathbf{k}\cdot\mathbf{r}} + e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}} \frac{e^{ikr}}{r} f(\hat{\mathbf{r}}),$$

where $\mathbf{r}' = \mathbf{r} - \mathbf{a}$, $r' = |\mathbf{r}'|$, $\hat{\mathbf{r}}' = \mathbf{r}'/r'$ and $k = |\mathbf{k}|$, $\mathbf{k}' = k\hat{\mathbf{r}}$. For electrons incident on N atoms in a regular Bravais crystal lattice show that the differential cross-section for scattering in the direction $\hat{\mathbf{r}}$ is

$$\frac{d\sigma}{d\Omega} = N\,|f(\hat{\mathbf{r}})|^2\,\Delta(\mathbf{k}-\mathbf{k}').$$

Derive an explicit form for $\Delta(\mathbf{Q})$ and show that it is strongly peaked when $\mathbf{Q} \approx \mathbf{b}$ for $\mathbf{b}$ a reciprocal lattice vector.

State the Born approximation for $f(\hat{\mathbf{r}})$ when the scattering is due to a potential $V(\mathbf{r})$. Calculate the Born approximation for the case $V(\mathbf{r}) = -a\,\delta(\mathbf{r})$.

Electrons with de Broglie wavelength λ are incident on a target composed of many randomly oriented small crystals. They are found to be scattered strongly through an angle of 60°. What is the likely distance between planes of atoms in the crystal responsible for the scattering?

Paper 1, Section II

27I Applied Probability

(a) Define what it means to say that π is an equilibrium distribution for a Markov chain on a countable state space with Q-matrix $Q = (q_{ij})$, and give an equation which is satisfied by any equilibrium distribution. Comment on the possible non-uniqueness of equilibrium distributions.

(b) State a theorem on convergence to an equilibrium distribution for a continuous-time Markov chain.

A continuous-time Markov chain $(X_t,\ t \geqslant 0)$ has three states 1, 2, 3 and the Q-matrix $Q = (q_{ij})$ is of the form

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1/2 & \lambda_1/2 \\ \lambda_2/2 & -\lambda_2 & \lambda_2/2 \\ \lambda_3/2 & \lambda_3/2 & -\lambda_3 \end{pmatrix},$$

where the rates $\lambda_1, \lambda_2, \lambda_3 \in [0, \infty)$ are not all zero.

[Note that some of the λ_i may be zero, and those cases may need special treatment.]

(c) Find the equilibrium distributions of the Markov chain in question. Specify the cases of uniqueness and non-uniqueness.

(d) Find the limit of the transition matrix $P(t) = \exp(tQ)$ when $t \to \infty$.

(e) Describe the jump chain (Y_n) and its equilibrium distributions. If $\widehat{P}$ is the jump probability matrix, find the limit of $\widehat{P}^n$ as $n \to \infty$.

Paper 2, Section II

27I Applied Probability

(a) Let S_k be the sum of k independent exponential random variables of rate $k\mu$. Compute the moment generating function $\phi_{S_k}(\theta) = \mathbb{E}e^{\theta S_k}$ of S_k. Show that, as $k \to \infty$, functions $\phi_{S_k}(\theta)$ converge to a limit. Describe the random variable S for which the limiting function $\lim_{k\to\infty} \phi_{S_k}(\theta)$ coincides with $\mathbb{E}e^{\theta S}$.

(b) Define the M/G/1 queue with infinite capacity (sometimes written M/G/1/∞). Introduce the embedded discrete-time Markov chain (X_n) and write down the recursive relation between X_n and X_{n-1}.

Consider, for each fixed k and for $0 < \lambda < \mu$, an M/G/1/∞ queue with arrival rate λ and with service times distributed as S_k. Assume that the queue is empty at time 0. Let T_k be the earliest time at which a customer departs leaving the queue empty. Let A be the first arrival time and $B_k = T_k - A$ the length of the busy period.

(c) Prove that the moment generating functions $\phi_{B_k}(\theta) = \mathbb{E}e^{\theta B_k}$ and $\phi_{S_k}(\theta)$ are related by the equation

$$\phi_{B_k}(\theta) = \phi_{S_k}\left(\theta - \lambda\big(1 - \phi_{B_k}(\theta)\big)\right),$$

(d) Prove that the moment generating functions $\phi_{T_k}(\theta) = \mathbb{E}e^{\theta T_k}$ and $\phi_{S_k}(\theta)$ are related by the equation

$$\frac{\lambda - \theta}{\lambda}\phi_{T_k}(\theta) = \phi_{S_k}\left((\lambda - \theta)\big(\phi_{T_k}(\theta) - 1\big)\right).$$

(e) Assume that, for all $\theta < \lambda$,

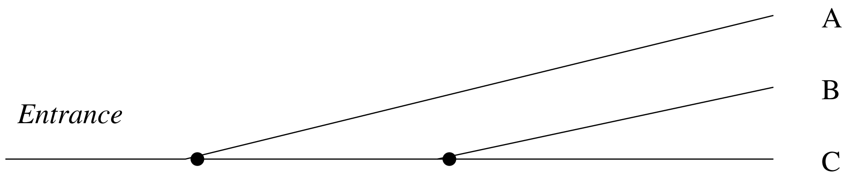
$$\lim_{k\to\infty} \phi_{B_k}(\theta) = \mathbb{E}e^{\theta B}, \quad \lim_{k\to\infty} \phi_{T_k}(\theta) = \mathbb{E}e^{\theta T},$$

for some random variables B and T. Calculate $\mathbb{E}B$ and $\mathbb{E}T$. What service time distribution do these values correspond to?

Paper 3, Section II

26I Applied Probability

Cars looking for a parking space are directed to one of three unlimited parking lots A, B and C. First, immediately after the entrance, the road forks: one direction is to lot A, the other to B and C. Shortly afterwards, the latter forks again, between B and C. See the diagram below.



The policeman at the first road fork directs an entering car with probability $1/3$ to A and with probability $2/3$ to the second fork. The policeman at the second fork sends the passing cars to B or C alternately: cars 1, 3, 5, ... approaching the second fork go to B and cars 2, 4, 6, ... to C.

Assuming that the total arrival process $(N(t))$ of cars is Poisson of rate λ, consider the processes $(X^{\rm A}(t))$, $(X^{\rm B}(t))$ and $(X^{\rm C}(t))$, $t \geqslant 0$, where $X^i(t)$ is the number of cars directed to lot i by time t, for $i =$A, B, C. The times for a car to travel from the first to the second fork, or from a fork to the parking lot, are all negligible.

(a) Characterise each of the processes $(X^{\rm A}(t))$, $(X^{\rm B}(t))$ and $(X^{\rm C}(t))$, by specifying if it is (i) Poisson, (ii) renewal or (iii) delayed renewal. Correspondingly, specify the rate, the holding-time distribution and the distribution of the delay.

(b) In the case of a renewal process, determine the equilibrium delay distribution.

(c) Given $s, t > 0$, write down explicit expressions for the probability $\mathbb{P}\big(X^i(s) = X^i(s+t)\big)$ that the interval $(s, t+s)$ is free of points in the corresponding process, $i = \mathrm{A}, \mathrm{B}, \mathrm{C}$.

Paper 4, Section II

26I Applied Probability

(a) Let (X_t) be an irreducible continuous-time Markov chain on a finite or countable state space. What does it mean to say that the chain is (i) transient, (ii) recurrent, (iii) positive recurrent, (iv) null recurrent? What is the relation between equilibrium distributions and properties (iii) and (iv)?

A population of microorganisms develops in continuous time; the size of the population is a Markov chain (X_t) with states 0, 1, 2, ... Suppose $X_t = n$. It is known that after a short time s, the probability that X_t increased by one is $\lambda(n+1)s + o(s)$ and (if $n \geqslant 1$) the probability that the population was exterminated between times t and $t+s$ and never revived by time $t+s$ is $\mu s + o(s)$. Here λ and μ are given positive constants. All other changes in the value of X_t have a combined probability $o(s)$.

(b) Write down the Q-matrix of Markov chain (X_t) and determine if (X_t) is irreducible. Show that (X_t) is non-explosive. Determine the jump chain.

(c) Now assume that

$$\mu = \lambda .$$

Determine whether the chain is transient or recurrent, and in the latter case whether it is positive or null recurrent. Answer the same questions for the jump chain. Justify your answers.

Paper 1, Section II

31C Asymptotic Methods

For $\lambda > 0$ let

$$I(\lambda) \;=\; \int_0^b f(x)\,\mathrm{e}^{-\lambda x}\,dx\,, \qquad \text{with} \quad 0 < b < \infty\,.$$

Assume that the function $f(x)$ is continuous on $0 < x \leqslant b$, and that

$$f(x) \;\sim\; x^{\alpha}\sum_{n=0}^{\infty} a_n\,x^{n\beta},$$

as $x \to 0_+$, where $\alpha > -1$ and $\beta > 0$.

(a) Explain briefly why in this case straightforward partial integrations in general cannot be applied for determining the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$.

(b) Derive with proof an asymptotic expansion for $I(\lambda)$ as $\lambda \to \infty$.

(c) For the function

$$B(s,t) = \int_0^1 u^{s-1}\,(1-u)^{t-1}\,du\,, \qquad s,t > 0\,,$$

obtain, using the substitution $u = e^{-x}$, the first two terms in an asymptotic expansion as $s \to \infty$. What happens as $t \to \infty$?

[*Hint: The following formula may be useful*

$$\Gamma(y) \;=\; \int_0^\infty x^{y-1}\,\mathrm{e}^{-x}\,dt\,, \qquad \text{for} \quad x > 0\,.$$
]

Paper 3, Section II

31C Asymptotic Methods

Consider the ordinary differential equation

$$y'' \;=\; (|x| - E)\,y\,,$$

subject to the boundary conditions $y(\pm\infty) = 0$. Write down the general form of the Liouville-Green solutions for this problem for $E > 0$ and show that asymptotically the eigenvalues E_n, $n \in \mathbb{N}$ and $E_n < E_{n+1}$, behave as $E_n = \mathrm{O}(n^{2/3})$ for large n.

Paper 4, Section II

31C Asymptotic Methods

(a) Consider for $\lambda > 0$ the Laplace type integral

$$I(\lambda) = \int_a^b f(t)\,\mathrm{e}^{-\lambda\phi(t)}\,dt\,,$$

for some finite $a, b \in \mathbb{R}$ and smooth, real-valued functions $f(t), \phi(t)$. Assume that the function $\phi(t)$ has a single minimum at $t = c$ with $a < c < b$. Give an account of Laplace's method for finding the leading order asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$ and briefly discuss the difference if instead $c = a$ or $c = b$, i.e. when the minimum is attained at the boundary.

(b) Determine the leading order asymptotic behaviour of

$$I(\lambda) = \int_{-2}^{1} \cos t\,\mathrm{e}^{-\lambda t^2}\,dt\,, \qquad (*)$$

as $\lambda \to \infty$.

(c) Determine also the leading order asymptotic behaviour when $\cos t$ is replaced by $\sin t$ in $(*)$.

Paper 1, Section I

9D Classical Dynamics

A system with coordinates q_i, $i = 1, \ldots, n$, has the Lagrangian $L(q_i, \dot{q}_i)$. Define the energy E.

Consider a charged particle, of mass m and charge e, moving with velocity $\mathbf{v}$ in the presence of a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. The usual vector equation of motion can be derived from the Lagrangian

$$L = \frac{1}{2} m\, \mathbf{v}^2 + e\, \mathbf{v} \cdot \mathbf{A}\,,$$

where $\mathbf{A}$ is the vector potential.

The particle moves in the presence of a field such that

$$\mathbf{A} = (0,\, r\, g(z),\, 0)\,, \qquad g(z) > 0\,,$$

referred to cylindrical polar coordinates (r, ϕ, z). Obtain two constants of the motion, and write down the Lagrangian equations of motion obtained by variation of r, ϕ and z.

Show that, if the particle is projected from the point (r_0, ϕ_0, z_0) with velocity $(0, -2\,(e/m)\, r_0\, g(z_0), 0)$, it will describe a circular orbit provided that $g'(z_0) = 0$.

Paper 2, Section I

9D Classical Dynamics

Given the form

$$T = \frac{1}{2}\, T_{ij}\, \dot{q}_i\, \dot{q}_j\,, \qquad V = \frac{1}{2}\, V_{ij}\, q_i\, q_j\,,$$

for the kinetic energy T and potential energy V of a mechanical system, deduce Lagrange's equations of motion.

A light elastic string of length $4b$, fixed at both ends, has three particles, each of mass m, attached at distances $b, 2b, 3b$ from one end. Gravity can be neglected. The particles vibrate with small oscillations transversely to the string, the tension S in the string providing the restoring force. Take the displacements of the particles, q_i, $i = 1, 2, 3$, to be the generalized coordinates. Take units such that $m = 1$, $S/b = 1$ and show that

$$V = \frac{1}{2}\left[{q_1}^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + {q_3}^2 \right].$$

Find the normal–mode frequencies for this system.

Paper 3, Section I

9D Classical Dynamics

Euler's equations for the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ of a rigid body, viewed in the body frame, are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\, \omega_2\, \omega_3$$

and cyclic permutations, where the principal moments of inertia are assumed to obey $I_1 < I_2 < I_3$.

Write down two quadratic first integrals of the motion.

There is a family of solutions $\boldsymbol{\omega}(t)$, unique up to time–translations $t \to (t - t_0)$, which obey the boundary conditions $\boldsymbol{\omega} \to (0, \Omega, 0)$ as $t \to -\infty$ and $\boldsymbol{\omega} \to (0, -\Omega, 0)$ as $t \to \infty$, for a given positive constant Ω. Show that, for such a solution, one has

$$\mathbf{L}^2 = 2EI_2\,,$$

where $\mathbf{L}$ is the angular momentum and E is the kinetic energy.

By eliminating ω_1 and ω_3 in favour of ω_2, or otherwise, show that, in this case, the second Euler equation reduces to

$$\frac{ds}{d\tau} = 1 - s^2\,,$$

where $s = \omega_2/\Omega$ and $\tau = \Omega t \left[(I_1 - I_2)(I_2 - I_3)/I_1 I_3\right]^{1/2}$. Find the general solution $s(\tau)$.

[You are not expected to calculate $\omega_1(t)$ or $\omega_3(t)$.]

Paper 4, Section I

9D Classical Dynamics

A system with one degree of freedom has Lagrangian $L(q, \dot{q})$. Define the canonical momentum p and the energy E. Show that E is constant along any classical path.

Consider a classical path $q_c(t)$ with the boundary–value data

$$q_c(0) = q_I\,, \qquad q_c(T) = q_F\,, \quad T > 0\,.$$

Define the action $S_c(q_I, q_F, T)$ of the path. Show that the total derivative dS_c/dT along the classical path obeys

$$\frac{dS_c}{dT} = L\,.$$

Using Lagrange's equations, or otherwise, deduce that

$$\frac{\partial S_c}{\partial q_F} = p_F\,, \qquad \frac{\partial S_c}{\partial T} = -E\,,$$

where p_F is the final momentum.

Paper 2, Section II

15D Classical Dynamics

An axially–symmetric top of mass m is free to rotate about a fixed point O on its axis. The principal moments of inertia about O are A, A, C, and the centre of gravity G is at a distance ℓ from O. Define Euler angles θ, ϕ and ψ which specify the orientation of the top, where θ is the inclination of OG to the upward vertical. Show that there are three conserved quantities for the motion, and give their physical meaning.

Initially, the top is spinning with angular velocity n about OG, with G vertically above O, before being disturbed slightly. Show that, in the subsequent motion, θ will remain close to zero provided $C^2n^2 > 4mg\ell A$, but that if $C^2n^2 < 4mg\ell A$, then θ will attain a maximum value given by

$$\cos\theta \simeq (C^2n^2/\,2mg\ell A) - 1\,.$$

Paper 4, Section II

15D Classical Dynamics

A system is described by the Hamiltonian $H(q,p)$. Define the *Poisson bracket* $\{f,g\}$ of two functions $f(q,p,t)$, $g(q,p,t)$, and show from Hamilton's equations that

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}\,.$$

Consider the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right),$$

and define

$$a = (p - i\omega q)/(2\omega)^{1/2}\,, \qquad a^* = (p + i\omega q)/(2\omega)^{1/2}\,,$$

where $i = \sqrt{-1}$. Evaluate $\{a,a\}$ and $\{a,a^*\}$, and show that $\{a,H\} = -i\omega a$ and $\{a^*,H\} = i\omega a^*$. Show further that, when $f(q,p,t)$ is regarded as a function of the independent complex variables a, a^* and of t, one has

$$\frac{df}{dt} = i\omega\left(a^*\frac{\partial f}{\partial a^*} - a\,\frac{\partial f}{\partial a}\right) + \frac{\partial f}{\partial t}\,.$$

Deduce that both $\log a^* - i\omega t$ and $\log a + i\omega t$ are constant during the motion.

Paper 1, Section I

4H Coding and Cryptography

Explain what is meant by saying that a binary code $\mathcal{C}$ is a decodable code with words C_j of length l_j for $1 \leqslant j \leqslant n$. Prove the MacMillan inequality which states that, for such a code,

$$\sum_{j=1}^{n} 2^{-l_j} \leqslant 1 \,.$$

Paper 2, Section I

4H Coding and Cryptography

Describe the standard Hamming code of length 7, proving that it corrects a single error. Find its weight enumeration polynomial.

Paper 3, Section I

4H Coding and Cryptography

What is a linear code? What is a parity check matrix for a linear code? What is the minimum distance $d(C)$ for a linear code C?

If C_1 and C_2 are linear codes having a certain relation (which you should specify), define the bar product $C_1 | \, C_2$. Show that

$$d(C_1|\, C_2) \;=\; \min\left\{2d(C_1), d(C_2)\right\} .$$

If C_1 has parity check matrix P_1 and C_2 has parity check matrix P_2, find a parity check matrix for $C_1|\, C_2$.

Paper 4, Section I

4H Coding and Cryptography

What is the discrete logarithm problem?

Describe the Diffie–Hellman key exchange system for two people. What is the connection with the discrete logarithm problem? Why might one use this scheme rather than just a public key system or a classical (pre-1960) coding system?

Extend the Diffie–Hellman system to n people using $n(n-1)$ transmitted numbers.

Paper 1, Section II

12H Coding and Cryptography

State and prove Shannon's theorem for the capacity of a noisy memoryless binary symmetric channel, defining the terms you use.

[You may make use of any form of Stirling's formula and any standard theorems from probability, provided that you state them exactly.]

Paper 2, Section II

12H Coding and Cryptography

The Van der Monde matrix $V(x_0, x_1, \ldots, x_{r-1})$ is the $r \times r$ matrix with (i, j)th entry x_{i-1}^{j-1}. Find an expression for $\det V(x_0, x_1, \ldots, x_{r-1})$ as a product. Explain why this expression holds if we work modulo p a prime.

Show that $\det V(x_0, x_1, \ldots, x_{r-1}) \equiv 0$ modulo p if $r > p$, and that there exist $x_0, \ldots, x_{p-1}$ such that $\det V(x_0, x_1, \ldots, x_{p-1}) \not\equiv 0$. By using Wilson's theorem, or otherwise, find the possible values of $\det V(x_0, x_1, \ldots, x_{p-1})$ modulo p.

The Dark Lord Y'Trinti has acquired the services of the dwarf Trigon who can engrave pairs of very large integers on very small rings. The Dark Lord wishes Trigon to engrave n rings in such a way that anyone who acquires r of the rings and knows the Prime Perilous p can deduce the Integer N of Power, but owning $r-1$ rings will give no information whatsoever. The integers N and p are very large and $p > N$. Advise the Dark Lord.

For reasons to be explained in the prequel, Trigon engraves an $(n+1)$st ring with random integers. A band of heroes (who know the Prime Perilous and all the information contained in this question) set out to recover the rings. What, if anything, can they say, with very high probability, about the Integer of Power if they have r rings (possibly including the fake)? What can they say if they have $r+1$ rings? What if they have $r+2$ rings?

Paper 1, Section I

10D Cosmology

What is meant by the expression 'Hubble time'?

For $a(t)$ the scale factor of the universe and assuming $a(0) = 0$ and $a(t_0) = 1$, where t_0 is the time now, obtain a formula for the size of the particle horizon R_0 of the universe.

Taking

$$a(t) = (t/t_0)^\alpha ,$$

show that R_0 is finite for certain values of α. What might be the physically relevant values of α? Show that the age of the universe is less than the Hubble time for these values of α.

Paper 2, Section I

10D Cosmology

The number density $n = N/V$ for a photon gas in equilibrium is given by

$$n = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2}{e^{h\nu/kT}-1}\,d\nu ,$$

where ν is the photon frequency. By letting $x = h\nu/kT$, show that

$$n = \alpha T^3 ,$$

where α is a constant which need not be evaluated.

The photon entropy density is given by

$$s = \beta T^3 ,$$

where β is a constant. By considering the entropy, explain why a photon gas cools as the universe expands.

Paper 3, Section I

10D Cosmology

Consider a homogenous and isotropic universe with mass density $\rho(t)$, pressure $P(t)$ and scale factor $a(t)$. As the universe expands its energy changes according to the relation $dE = -PdV$. Use this to derive the fluid equation

$$\dot{\rho} = -3\,\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right).$$

Use conservation of energy applied to a test particle at the boundary of a spherical fluid element to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi}{3}\,G\rho - \frac{k}{a^2}\,c^2\,,$$

where k is a constant. State any assumption you have made. Briefly state the significance of k.

Paper 4, Section I

10D Cosmology

The linearised equation for the growth of density perturbations, $\delta_{\mathbf{k}}$, in an isotropic and homogenous universe is

$$\ddot{\delta}_{\mathbf{k}} + 2\,\frac{\dot{a}}{a}\,\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^2\,\mathbf{k}^2}{a^2} - 4\pi\,G\rho\right)\delta_{\mathbf{k}} = 0\,,$$

where ρ is the density of matter, c_s the sound speed, $c_s^2 = dP/d\rho$, and $\mathbf{k}$ is the comoving wavevector and $a(t)$ is the scale factor of the universe.

What is the Jean's length? Discuss its significance for the growth of perturbations.

Consider a universe filled with pressure-free matter with $a(t) = (t/t_0)^{2/3}$. Compute the resulting equation for the growth of density perturbations. Show that your equation has growing and decaying modes and comment briefly on the significance of this fact.

Paper 1, Section II

15D Cosmology

A star has pressure $P(r)$ and mass density $\rho(r)$, where r is the distance from the centre of the star. These quantities are related by the pressure support equation

$$P' = -\frac{Gm\rho}{r^2},$$

where $P' = dP/dr$ and $m(r)$ is the mass within radius r. Use this to derive the virial theorem

$$E_{\text{grav}} = -3\langle P\rangle V,$$

where E_{grav} is the total gravitational potential energy and $\langle P\rangle$ the average pressure.

The total kinetic energy of a spherically symmetric star is related to $\langle P\rangle$ by

$$E_{\text{kin}} = \alpha\langle P\rangle V,$$

where α is a constant. Use the virial theorem to determine the condition on α for gravitational binding. By considering the relation between pressure and 'internal energy' U for an ideal gas, determine α for the cases of a) an ideal gas of non-relativistic particles, b) an ideal gas of ultra-relativistic particles.

Why does your result imply a maximum mass for any star? Briefly explain what is meant by the Chandrasekhar limit.

A white dwarf is in orbit with a companion star. It slowly accretes matter from the other star until its mass exceeds the Chandrasekhar limit. Briefly explain its subsequent evolution.

Paper 3, Section II

15D Cosmology

The number density for particles in thermal equilibrium, neglecting quantum effects, is

$$n = g_s \frac{4\pi}{h^3} \int p^2 \, dp \, \exp(-(E(p) - \mu)/kT) \,,$$

where g_s is the number of degrees of freedom for the particle with energy $E(p)$ and μ is its chemical potential. Evaluate n for a non-relativistic particle.

Thermal equilibrium between two species of non-relativistic particles is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta \,,$$

where α and β are massless particles. Evaluate the ratio of number densities n_a/n_b given that their respective masses are m_a and m_b and chemical potentials are μ_a and μ_b.

Explain how a reaction like the one above is relevant to the determination of the neutron to proton ratio in the early universe. Why does this ratio not fall rapidly to zero as the universe cools?

Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei. Letting

$$Y_{He} = \rho_{He}/\rho$$

be the fraction of the universe's helium, compute Y_{He} as a function of the ratio $r = n_n/n_p$ at the time of nucleosynthesis.

Paper 1, Section II
25H Differential Geometry

(i) State the definition of *smooth manifold with boundary* and define the notion of *boundary.* Show that the boundary ∂X is a manifold (without boundary) with $\dim \partial X = \dim X - 1$.

(ii) Let $0 < a < 1$ and let x_1, x_2, x_3, x_4 denote Euclidean coordinates on $\mathbb{R}^4$. Show that the set

$$X = \{x_1^2+x_2^2+x_3^2-x_4^2 \leqslant a\} \cap \{x_1^2+x_2^2+x_3^2+x_4^2 = 1\} \cap \{x_1^2+2\,x_2^2+x_3^2+x_4^2 = 3/2\}$$

is a manifold with boundary and compute its dimension. You may appeal to standard results concerning regular values of smooth functions.

(iii) Determine if the following statements are true or false, giving reasons:

a. If X and Y are manifolds, $f : X \to Y$ smooth and $Z \subset Y$ a submanifold of codimension r such that f is not transversal to Z, then $f^{-1}(Z)$ is not a submanifold of codimension r in X.

b. If X and Y are manifolds and $f : X \to Y$ is smooth, then the set of regular values of f is open in Y.

c. If X and Y are manifolds and $f : X \to Y$ is smooth then the set of critical points is of measure 0 in X.

Paper 2, Section II
25H Differential Geometry

(i) State and prove the isoperimetric inequality for plane curves. You may appeal to Wirtinger's inequality as long as you state it precisely.

(ii) State Fenchel's theorem for curves in space.

(iii) Let $\alpha : I \to \mathbb{R}^2$ be a closed regular plane curve bounding a region K. Suppose $K \supset [p_1, p_1 + d_1] \times [p_2, p_2 + d_2]$, for $d_1 > 0$, $d_2 > 0$, i.e. K contains a rectangle of dimensions d_1, d_2. Let $k(s)$ denote the signed curvature of α with respect to the inward pointing normal, where α is parametrised anticlockwise. Show that there exists an $s_0 \in I$ such that $k(s_0) \leqslant \sqrt{\pi/(d_1 d_2)}$.

Paper 3, Section II

24H Differential Geometry

(i) State and prove the *Theorema Egregium.*

(ii) Define the notions *principal curvatures*, *principal directions* and *umbilical point.*

(iii) Let $S \subset \mathbb{R}^3$ be a connected compact regular surface (without boundary), and let $D \subset S$ be a dense subset of S with the following property. For all $p \in D$, there exists an open neighbourhood $\mathcal{U}_p$ of p in S such that for all $\theta \in [0, 2\pi)$, $\psi_{p,\theta}(\mathcal{U}_p) = \mathcal{U}_p$, where $\psi_{p,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes rotation by θ around the line through p perpendicular to $T_p S$. Show that S is in fact a sphere.

Paper 4, Section II

24H Differential Geometry

(i) Let $S \subset \mathbb{R}^3$ be a regular surface. Define the notions *exponential map*, *geodesic polar coordinates*, *geodesic circles.*

(ii) State and prove *Gauss' lemma.*

(iii) Let S be a regular surface. For fixed $r > 0$, and points p, q in S, let $S_r(p)$, $S_r(q)$ denote the geodesic circles around p, q, respectively, of radius r. Show the following statement: for each $p \in S$, there exists an $r = r(p) > 0$ and a neighborhood $\mathcal{U}_p$ containing p such that for all $q \in \mathcal{U}_p$, the sets $S_r(p)$ and $S_r(q)$ are smooth 1-dimensional manifolds which intersect transversally. What is the cardinality mod 2 of $S_r(p) \cap S_r(q)$?

Paper 1, Section I

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = -\mu x + y\,, \qquad \dot{y} = \frac{x^2}{1+x^2} - \nu y\,,$$

where $x(t)$ and $y(t)$ are non-negative, the parameters μ and ν are strictly positive and $\mu \neq \nu$. Sketch the nullclines in the x, y plane. Deduce that for $\mu < \mu_c$ (where μ_c is to be determined) there are three fixed points. Find them and determine their type.

Sketch the phase portrait for $\mu < \mu_c$ and identify, qualitatively on your sketch, the stable and unstable manifolds of the saddle point. What is the final outcome of this system?

Paper 2, Section I

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = \mu\left(\frac{1}{3}x^3 - x\right) + y\,, \qquad \dot{y} = -x\,,$$

where the parameter $\mu > 0$. Using Lyapunov's approach, discuss the stability of the fixed point and its domain of attraction. Relevant definitions or theorems that you use should be stated carefully, but proofs are not required.

Paper 3, Section I

7D Dynamical Systems

Let $I = [0, 1)$. The sawtooth (Bernoulli shift) map $F : I \to I$ is defined by

$$F(x) = 2x \ [\bmod 1]\,.$$

Describe the effect of F using binary notation. Show that F is continuous on I except at $x = \frac{1}{2}$. Show also that F has N-periodic points for all $N \geqslant 2$. Are they stable?

Explain why F is chaotic, using Glendinning's definition.

Paper 4, Section I

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = y + \frac{1}{4}x\left(1 - 2x^2 - 2y^2\right), \qquad \dot{y} = -x + \frac{1}{2}y\left(1 - x^2 - y^2\right).$$

Use the Poincaré–Bendixson theorem, which should be stated carefully, to obtain a domain $\mathcal{D}$ in the xy-plane, within which there is at least one periodic orbit.

Paper 3, Section II

14D Dynamical Systems

Describe informally the concepts of extended stable manifold theory. Illustrate your discussion by considering the 2-dimensional flow

$$\dot{x} = \mu x + xy - x^3, \qquad \dot{y} = -y + y^2 - x^2,$$

where μ is a parameter with $|\mu| \ll 1$, in a neighbourhood of the origin. Determine the nature of the bifurcation.

Paper 4, Section II

14D Dynamical Systems

Let $I = [0,1]$ and consider continuous maps $F : I \to I$. Give an informal outline description of the two different bifurcations of fixed points of F that can occur.

Illustrate your discussion by considering in detail the logistic map

$$F(x) = \mu x(1-x),$$

for $\mu \in (0, 1+\sqrt{6}]$.

Describe qualitatively what happens for $\mu \in (1+\sqrt{6}, 4]$.

[You may assume without proof that

$$x - F^2(x) = x(\mu x - \mu + 1)(\mu^2 x^2 - \mu(\mu+1)x + \mu + 1).$$

]

Paper 1, Section II

35B Electrodynamics

The vector potential A^{μ} is determined by a current density distribution j^{μ} in the gauge $\partial_{\mu} A^{\mu} = 0$ by

$$\Box A^{\mu} = -\mu_0 j^{\mu}, \qquad \Box = -\frac{\partial^2}{\partial t^2} + \nabla^2,$$

in units where $c = 1$.

Describe how to justify the result

$$A^{\mu}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{j^{\mu}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}, \qquad t' = t - |\mathbf{x} - \mathbf{x}'|.$$

A plane square loop of thin wire, edge lengths l, has its centre at the origin and lies in the (x, y) plane. For $t < 0$, no current is flowing in the loop, but at $t = 0$ a constant current I is turned on.

Find the vector potential at the point $(0, 0, z)$ as a function of time due to a single edge of the loop.

What is the electric field due to the entire loop at $(0, 0, z)$ as a function of time? Give a careful justification of your answer.

Paper 3, Section II

36B Electrodynamics

A particle of rest-mass m, electric charge q, is moving relativistically along the path $x^\mu(s)$ where s parametrises the path.

Write down an action for which the extremum determines the particle's equation of motion in an electromagnetic field given by the potential $A^\mu(x)$.

Use your action to derive the particle's equation of motion in a form where s is the proper time.

Suppose that the electric and magnetic fields are given by

$$\begin{aligned}\mathbf{E} &= (0,0,E)\,,\\ \mathbf{B} &= (0,B,0)\,.\end{aligned}$$

where E and B are constants and $B > E > 0$.

Find $x^\mu(s)$ given that the particle starts at rest at the origin when $s = 0$.

Describe qualitatively the motion of the particle.

Paper 4, Section II

35B Electrodynamics

In a superconductor the number density of charge carriers of charge q is n_s . Suppose that there is a time-independent magnetic field described by the three-vector potential $\mathbf{A}$.

Derive an expression for the superconducting current.

Explain how your answer is gauge invariant.

Suppose that for $z < 0$ there is a constant magnetic field $\mathbf{B}_0$ in a vacuum and, for $z > 0$, there is a uniform superconductor. Derive the magnetic field for $z > 0$.

Paper 1, Section II

37A Fluid Dynamics II

Write down the Navier-Stokes equation for the velocity $\mathbf{u}(\mathbf{x}, t)$ of an incompressible viscous fluid of density ρ and kinematic viscosity ν. Cast the equation into dimensionless form. Define *rectilinear flow*, and explain why the spatial form of any steady rectilinear flow is independent of the Reynolds number.

(i) Such a fluid is contained between two infinitely long plates at $y = 0, y = a$. The lower plate is at rest while the upper plate moves at constant speed U in the x direction. There is an applied pressure gradient $dp/dx = -G\rho\nu$ in the x direction. Determine the flow field.

(ii) Now there is no applied pressure gradient, but baffles are attached to the lower plate at a distance L from each other ($L \gg a$), lying between the plates so as to prevent any net volume flux in the x direction. Assuming that far from the baffles the flow is essentially rectilinear, determine the flow field and the pressure gradient in the fluid.

Paper 2, Section II

37A Fluid Dynamics II

What is lubrication theory? Explain the assumptions that go into the theory.

Viscous fluid with dynamic viscosity μ and density ρ is contained between two flat plates, which approach each other at uniform speed V. The first is fixed at $y = 0$, $-L < x < L$. The second has its ends at $(-L,\, h_0 - \Delta h - Vt)$, $(L,\, h_0 + \Delta h - Vt)$, where $\Delta h \sim h_0 \ll L$. There is no flow in the z direction, and all variation in z may be neglected. There is no applied pressure gradient in the x direction.

Assuming that V is so small that lubrication theory applies, derive an expression for the horizontal volume flux $Q(x)$ at $t = 0$, in terms of the pressure gradient. Show that mass conservation implies that $dQ/dx = V$, so that $Q(L) - Q(-L) = 2VL$. Derive another relation between $Q(L)$ and $Q(-L)$ by setting the pressures at $x = \pm L$ to be equal, and hence show that

$$Q(\pm L) = VL\left(\frac{\Delta h}{h_0} \pm 1\right).$$

Show that lubrication theory applies if $V \ll \mu/h_0\rho$.

Paper 3, Section II

37A Fluid Dynamics II

The equation for the vorticity $\omega(x,y)$ in two-dimensional incompressible flow takes the form

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right),$$

where

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right),$$

and $\psi(x,y)$ is the stream function.

Show that this equation has a time-dependent similarity solution of the form

$$\psi = CxH(t)^{-1}\phi(\eta), \qquad \omega = -CxH(t)^{-3}\phi_{\eta\eta}(\eta) \quad \text{for} \quad \eta = yH(t)^{-1},$$

if $H(t) = \sqrt{2Ct}$ and ϕ satisfies the equation

$$3\,\phi_{\eta\eta} + \eta\phi_{\eta\eta\eta} - \phi_\eta\phi_{\eta\eta} + \phi\phi_{\eta\eta\eta} + \frac{1}{R}\,\phi_{\eta\eta\eta\eta} = 0\,, \qquad (*)$$

and $R = C/\nu$ is the effective Reynolds number.

Show that this solution is appropriate for the problem of two-dimensional flow between the rigid planes $y = \pm H(t)$, and determine the boundary conditions on ϕ in that case.

Verify that $(*)$ has exact solutions, satisfying the boundary conditions, of the form

$$\phi = \frac{(-1)^k}{k\pi}\sin(k\pi\eta) - \eta\,, \qquad k = 1,2,\ldots\ ,$$

when $R = k^2\pi^2/4$. Sketch this solution when k is large, and discuss whether such solutions are likely to be realised in practice.

Paper 4, Section II

37A Fluid Dynamics II

An axisymmetric incompressible Stokes flow has the Stokes stream function $\Psi(R,\theta)$ in spherical polar coordinates (R,θ,ϕ). Give expressions for the components u_R, u_θ of the flow field in terms of Ψ. Show that the equation satisfied by Ψ is

$$\mathcal{D}^2(\mathcal{D}^2\Psi) = 0\,, \quad \text{where} \quad \mathcal{D}^2 = \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \qquad (*)$$

Fluid is contained between the two spheres $R = a, R = b$, with $b \gg a$. The fluid velocity vanishes on the outer sphere, while on the inner sphere $u_R = U\cos\theta, u_\theta = 0$. It is assumed that Stokes flow applies.

(i) Show that the Stokes stream function,

$$\Psi(R,\theta) = a^2 U \sin^2\theta \left(A\left(\frac{a}{R}\right) + B\left(\frac{R}{a}\right) + C\left(\frac{R}{a}\right)^2 + D\left(\frac{R}{a}\right)^4 \right),$$

is the general solution of $(*)$ proportional to $\sin^2\theta$ and write down the conditions on A, B, C, D that allow all the boundary conditions to be satisfied.

(ii) Now let $b \to \infty$, with $|\mathbf{u}| \to 0$ as $R \to \infty$. Show that $A = B = 1/4$ with $C = D = 0$.

(iii) Show that when b/a is very large but finite, then the coefficients have the approximate form

$$C \approx -\frac{3}{8}\frac{a}{b}, \quad D \approx \frac{1}{8}\frac{a^3}{b^3}, \quad A \approx \frac{1}{4} - \frac{3}{16}\frac{a}{b}, \quad B \approx \frac{1}{4} + \frac{9}{16}\frac{a}{b}.$$

Paper 1, Section I

8E Further Complex Methods

Let the complex-valued function $f(z)$ be analytic in the neighbourhood of the point z_0 and let $u(x,y)$ be the real part of $f(z)$. Show that

$$f(z) \;=\; 2\,u\!\left(\frac{z+\bar{z}_0}{2}\,,\,\frac{z-\bar{z}_0}{2i}\right) - \overline{f(z_0)}\,, \qquad z \;=\; x+iy\,.$$

Hence find the analytic function whose real part is

$$e^{-y}[\,x\cos x - y\sin x\,]\,.$$

Paper 2, Section I

8E Further Complex Methods

Define

$$F^{\pm}(x) \;=\; \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-(x\pm i\epsilon)}\,dt\,, \qquad x \,\in\, \mathbb{R}\,.$$

Using the fact that

$$F^{\pm}(x) \;=\; \pm\,\frac{f(x)}{2} + \frac{1}{2\pi i}\,P\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,dt\,, \qquad x \,\in\, \mathbb{R}\,,$$

where P denotes the Cauchy principal value, find two complex-valued functions $F^{+}(z)$ and $F^{-}(z)$ which satisfy the following conditions

1. $F^{+}(z)$ and $F^{-}(z)$ are analytic for $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ respectively, $z = x+iy$;
2. $F^{+}(x) - F^{-}(x) \;=\; \dfrac{\sin x}{x}\,, \qquad x \in \mathbb{R}\,;$
3. $F^{\pm}(z) = \mathrm{O}\left(\frac{1}{z}\right), \qquad z \to \infty\,, \qquad \operatorname{Im} z \neq 0\,.$

Paper 3, Section I

8E Further Complex Methods

Let $\Gamma(z)$ and $\zeta(z)$ denote the gamma and the zeta functions respectively, namely

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx\,, \qquad \operatorname{Re} z > 0\,,$$
$$\zeta(z) = \sum_{m=1}^{\infty} \frac{1}{m^z}\,, \qquad \operatorname{Re} z > 1\,.$$

By employing a series expansion of $(1-e^{-x})^{-2}$, prove the following identity

$$\int_0^\infty \frac{x^z}{(e^x-1)^2}\, dx = \Gamma(z+1)\big[\zeta(z) - \zeta(z+1)\big]\,, \qquad \operatorname{Re} z > 1\,.$$

Paper 4, Section I

8E Further Complex Methods

The hypergeometric function $F(a,b;c;z)$ can be expressed in the form

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt\,,$$

for appropriate restrictions on c, b, z.

Express the following integral in terms of a combination of hypergeometric functions

$$I(u,A) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{it(u+1)}}{e^{it}+iA}\, dt\,, \qquad |A| > 1\,.$$

[You may use without proof that $\Gamma(z+1) = z\Gamma(z)$.]

Paper 1, Section II

14E Further Complex Methods

Consider the partial differential equation for $u(x,t)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \beta\,\frac{\partial u}{\partial x}\,, \quad \beta > 0\,, \quad 0 < x < \infty\,, \quad t > 0\,, \tag{$*$}$$

where $u(x,t)$ is required to vanish rapidly for all t as $x \to \infty$.

(i) Verify that the PDE $(*)$ can be written in the following form

$$\left(e^{-ikx+(k^2-i\beta k)t}u\right)_t = \left(e^{-ikx+(k^2-i\beta k)t}\big[(ik+\beta)u + u_x\big]\right)_x .$$

(ii) Define $\hat{u}(k,t) = \int_0^\infty e^{-ikx}\,u(x,t)\,dx$, which is analytic for $\operatorname{Im} k \leqslant 0$. Determine $\hat{u}(k,t)$ in terms of $\hat{u}(k,0)$ and also the functions f_0, f_1 defined by

$$f_0(\omega,t) = \int_0^t e^{-\omega(t-t')}\,u(0,t')\,dt'\,, \qquad f_1(\omega,t) = \int_0^t e^{-\omega(t-t')}\,u_x(0,t')\,dt'\,.$$

(iii) Show that in the inverse transform expression for $u(x,t)$ the integrals involving f_0, f_1 may be transformed to the contour

$$L = \left\{k \in \mathbb{C} : \operatorname{Re}(k^2 - i\beta k) = 0,\ \operatorname{Im} k \geqslant \beta\right\}.$$

By considering $\hat{u}(k',t)$ where $k' = -k + i\beta$ and $k \in L$, show that it is possible to obtain an equation which allows f_1 to be eliminated.

(iv) Obtain an integral expression for the solution of $(*)$ subject to the the initial-boundary value conditions of given $u(x,0)$, $u(0,t)$.

[You need to show that

$$\int_L e^{ikx}\,\hat{u}(k',t)\,dk = 0\,, \quad x > 0\,,$$

by an appropriate closure of the contour which should be justified.]

Paper 2, Section II

14E Further Complex Methods

Let

$$I(z) = i \oint_C \frac{u^{z-1}}{u^2 - 4u + 1}\, du\,,$$

where C is a closed anti-clockwise contour which consists of the unit circle joined to a loop around a branch cut along the negative axis between -1 and 0. Show that

$$I(z) = F(z) + G(z)\,,$$

where

$$F(z) \;=\; 2\sin(\pi z) \int_0^1 \frac{x^{z-1}}{x^2 + 4x + 1}\, dx\,, \qquad \operatorname{Re} z \;>\; 0\,,$$

and

$$G(z) \;=\; \frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{i(z-1)\theta}}{1 + 2\sin^2\frac{\theta}{2}}\, d\theta\,, \qquad z \;\in\; \mathbb{C}\,.$$

Evaluate $I(z)$ using Cauchy's theorem. Explain how this may be used to obtain an analytic continuation of $F(z)$ valid for all $z \in \mathbb{C}$.

Paper 1, Section II

18H Galois Theory

Let $\mathbb{F}_q$ be a finite field with q elements and $\overline{\mathbb{F}}_q$ its algebraic closure.

(i) Give a non-zero polynomial $P(X)$ in $\mathbb{F}_q\,[X_1, \ldots, X_n]$ such that

$$P(\alpha_1, \ldots, \alpha_n) = 0 \qquad \text{for all } \alpha_1, \ldots, \alpha_n \in \mathbb{F}_q\,.$$

(ii) Show that every irreducible polynomial $P(X)$ of degree $n > 0$ in $\mathbb{F}_q[X]$ can be factored in $\overline{\mathbb{F}}_q[X]$ as $\big(X-\alpha\big)\big(X-\alpha^{q}\big)\big(X-\alpha^{q^2}\big)\cdots\big(X-\alpha^{q^{n-1}}\big)$ for some $\alpha \in \overline{\mathbb{F}}_q$. What is the splitting field and the Galois group of P over $\mathbb{F}_q$?

(iii) Let n be a positive integer and $\Phi_n(X)$ be the n-th cyclotomic polynomial. Recall that if K is a field of characteristic prime to n, then the set of all roots of Φ_n in K is precisely the set of all primitive n-th roots of unity in K. Using this fact, prove that if p is a prime number not dividing n, then p divides $\Phi_n(x)$ in $\mathbb{Z}$ for some $x \in \mathbb{Z}$ if and only if $p = an + 1$ for some integer a. Write down Φ_n explicitly for three different values of n larger than 2, and give an example of x and p as above for each n.

Paper 2, Section II

18H Galois Theory

(1) Let $F = \mathbb{Q}(\sqrt[3]{5}, \sqrt{5}, i)$. What is the degree of $F/\mathbb{Q}$? Justify your answer.

(2) Let F be a splitting field of $X^4 - 5$ over $\mathbb{Q}$. Determine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$. Determine all the subextensions of $F/\mathbb{Q}$, expressing each in the form $\mathbb{Q}(x)$ or $\mathbb{Q}(x, y)$ for some $x, y \in F$.

[*Hint: If an automorphism ρ of a field X has order 2, then for every $x \in X$ the element $x + \rho(x)$ is fixed by ρ.*]

Paper 3, Section II

18H Galois Theory

Let K be a field of characteristic 0. It is known that soluble extensions of K are contained in a succession of cyclotomic and Kummer extensions. We will refine this statement.

Let n be a positive integer. The n-th cyclotomic field over a field K is denoted by $K(\boldsymbol{\mu}_n)$. Let ζ_n be a primitive n-th root of unity in $K(\boldsymbol{\mu}_n)$.

(i) Write $\zeta_3 \in \mathbb{Q}(\boldsymbol{\mu}_3)$, $\zeta_5 \in \mathbb{Q}(\boldsymbol{\mu}_5)$ in terms of radicals. Write $\mathbb{Q}(\boldsymbol{\mu}_3)/\mathbb{Q}$ and $\mathbb{Q}(\boldsymbol{\mu}_5)/\mathbb{Q}$ as a succession of Kummer extensions.

(ii) Let $n > 1$, and $F := K(\zeta_1, \zeta_2, \ldots, \zeta_{n-1})$. Show that $F(\boldsymbol{\mu}_n)/F$ can be written as a succession of Kummer extensions, using the structure theorem of finite abelian groups (in other words, roots of unity can be written in terms of radicals). Show that every soluble extension of K is contained in a succession of Kummer extensions.

Paper 4, Section II

18H Galois Theory

Let K be a field of characteristic $\neq 2, 3$, and assume that K contains a primitive cubic root of unity ζ. Let $P \in K[X]$ be an irreducible cubic polynomial, and let α, β, γ be its roots in the splitting field F of P over K. Recall that the Lagrange resolvent x of P is defined as $x = \alpha + \zeta\beta + \zeta^2\gamma$.

(i) List the possibilities for the group $\mathrm{Gal}(F/K)$, and write out the set $\{\sigma(x) \mid \sigma \in \mathrm{Gal}(F/K)\}$ in each case.

(ii) Let $y = \alpha + \zeta\gamma + \zeta^2\beta$. Explain why x^3, y^3 must be roots of a quadratic polynomial in $K[X]$. Compute this polynomial for $P = X^3 + bX + c$, and deduce the criterion to identify $\mathrm{Gal}(F/K)$ through the element $-4b^3 - 27c^2$ of K.

Paper 1, Section II

36B General Relativity

Consider a spacetime $\mathcal{M}$ with a metric $g_{ab}(x^c)$ and a corresponding connection Γ^a_{bc}. Write down the differential equation satisfied by a geodesic $x^a(\lambda)$, where λ is an affine parameter.

Show how the requirement that

$$\delta \int g_{ab}(x^c)\, \frac{d}{d\lambda}x^a(\lambda)\, \frac{d}{d\lambda}x^b(\lambda)\; d\lambda \;=\; 0\;,$$

where δ denotes variation of the path, gives the geodesic equation and determines Γ^a_{bc}.

Show that the timelike geodesics for the 2–manifold with line element

$$ds^2 \;=\; t^{-2}\,(dx^2 - dt^2)$$

are given by

$$t^2 \;=\; x^2 + \alpha\, x + \beta\,,$$

where α and β are constants.

Paper 2, Section II

36B General Relativity

A vector field k^a which satisfies

$$k_{a;b} + k_{b;a} = 0$$

is called a Killing vector field. Prove that k^a is a Killing vector field if and only if

$$k^c g_{ab,c} + k^c{}_{,b}\, g_{ac} + k^c{}_{,a}\, g_{bc} = 0\,.$$

Prove also that if V^a satisfies

$$V^a{}_{;b} V^b = 0\,,$$

then

$$(V^a k_a)_{,b} V^b = 0 \qquad (*)$$

for any Killing vector field k^a.

In the two–dimensional space–time with coordinates $x^a = (u, v)$ and line element

$$ds^2 = -du^2 + u^2 dv^2\,,$$

verify that $(0, 1)$, $e^{-v}(1, u^{-1})$ and $e^v(-1, u^{-1})$ are Killing vector fields. Show, by using $(*)$ with V^a the tangent vector to a geodesic, that geodesics in this space–time are given by

$$\alpha e^v + \beta e^{-v} = 2\,\gamma u^{-1}\,,$$

where α, β and γ are arbitrary real constants.

Paper 4, Section II

36B General Relativity

The Schwarzschild line element is given by

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,,$$

where $F = 1 - r_s/r$ and r_s is the Schwarzschild radius. Obtain the equation of geodesic motion of photons moving in the equatorial plane, $\theta = \pi/2$, in the form

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \frac{h^2 F}{r^2} \,,$$

where τ is proper time, and E and h are constants whose physical significance should be indicated briefly.

Defining $u = 1/r$ show that light rays are determined by

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{1}{b}\right)^2 - u^2 + r_s \, u^3 \,,$$

where $b = h/E$ and r_s may be taken to be small. Show that, to zeroth order in r_s, a light ray is a straight line passing at distance b from the origin. Show that, to first order in r_s, the light ray is deflected through an angle $2r_s/b$. Comment briefly on some observational evidence for the result.

Paper 1, Section I

3F Geometry of Group Actions

Explain what it means to say that G is a *crystallographic group* of isometries of the Euclidean plane and that $\overline{G}$ is its *point group*. Prove the *crystallographic restriction:* a rotation in such a point group $\overline{G}$ must have order $1, 2, 3, 4$ or 6.

Paper 2, Section I

3F Geometry of Group Actions

Show that a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry for the Euclidean metric on the plane $\mathbb{R}^2$ if and only if there is a vector $\boldsymbol{v} \in \mathbb{R}^2$ and an orthogonal linear map $B \in \mathrm{O}(2)$ with

$$T(\boldsymbol{x}) = B(\boldsymbol{x}) + \boldsymbol{v} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^2 .$$

When T is an isometry with $\det B = -1$, show that T is either a reflection or a glide reflection.

Paper 3, Section I

3F Geometry of Group Actions

Let U be a "triangular" region in the unit disc $\mathbb{D}$ bounded by three hyperbolic geodesics $\gamma_1, \gamma_2, \gamma_3$ that do not meet in $\mathbb{D}$ nor on its boundary. Let J_k be inversion in γ_k and set

$$A = J_2 \circ J_1 ; \qquad B = J_3 \circ J_2 .$$

Let G be the group generated by the Möbius transformations A and B. Describe briefly a fundamental set for the group G acting on $\mathbb{D}$.

Prove that G is a free group on the two generators A and B. Describe the quotient surface $\mathbb{D}/G$.

Paper 4, Section I

3F Geometry of Group Actions

Define *loxodromic transformations* and explain how to determine when a Möbius transformation

$$T : z \mapsto \frac{az+b}{cz+d} \qquad \text{with} \quad ad - bc = 1$$

is loxodromic.

Show that any Möbius transformation that maps a disc Δ onto itself cannot be loxodromic.

Paper 1, Section II

11F Geometry of Group Actions

For which circles Γ does inversion in Γ interchange 0 and ∞?

Let Γ be a circle that lies entirely within the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let K be inversion in this circle Γ, let J be inversion in the unit circle, and let T be the Möbius transformation $K \circ J$. Show that, if z_0 is a fixed point of T, then

$$J(z_0) = K(z_0)$$

and this point is another fixed point of T.

By applying a suitable isometry of the hyperbolic plane $\mathbb{D}$, or otherwise, show that Γ is the set of points at a fixed hyperbolic distance from some point of $\mathbb{D}$.

Paper 4, Section II

12F Geometry of Group Actions

Explain briefly how Möbius transformations of the Riemann sphere are extended to give isometries of the unit ball $B^3 \subset \mathbb{R}^3$ for the hyperbolic metric.

Which Möbius transformations have extensions that fix the origin in B^3?

For which Möbius transformations T can we find a hyperbolic line in B^3 that T maps onto itself? For which of these Möbius transformations is there only one such hyperbolic line?

Paper 1, Section II

17F Graph Theory

(a) Define the *Ramsey number* $R(s)$. Show that for all integers $s \geqslant 2$ the Ramsey number $R(s)$ exists and that $R(s) \leqslant 4^s$.

(b) For any graph G, let $R(G)$ denote the least positive integer n such that in any red-blue colouring of the edges of the complete graph K_n there must be a monochromatic copy of G.

(i) How do we know that $R(G)$ exists for every graph G?

(ii) Let s be a positive integer. Show that, whenever the edge of K_{2s} are red-blue coloured, there must be a monochromatic copy of the complete bipartite graph $K_{1,s}$.

(iii) Suppose s is odd. By exhibiting a suitable colouring of K_{2s-1}, show that $R(K_{1,s}) = 2s$.

(iv) Suppose instead s is even. What is $R(K_{1,s})$? Justify your answer.

Paper 2, Section II

17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G contains a *matching* from X to Y?

State and prove Hall's Marriage Theorem, giving a necessary and sufficient condition for G to contain a matching from X to Y.

Now assume that G does contain a matching (from X to Y). For a subset $A \subset X$, $\Gamma(A)$ denotes the set of vertices adjacent to some vertex in A.

(i) Suppose $|\Gamma(A)| > |A|$ for every $A \subset X$ with $A \neq \emptyset$, X. Show that every edge of G is contained in a matching.

(ii) Suppose that every edge of G is contained in a matching and that G is connected. Show that $|\Gamma(A)| > |A|$ for every $A \subset X$ with $A \neq \emptyset$, X.

(iii) For each $n \geqslant 2$, give an example of G with $|X| = n$ such that every edge is contained in a matching but $|\Gamma(A)| = |A|$ for some $A \subset X$ with $A \neq \emptyset$, X.

(iv) Suppose that every edge of G is contained in a matching. Must every pair of independent edges in G be contained in a matching? Give a proof or counterexample as appropriate.

[No form of Menger's Theorem or of the Max-Flow-Min-Cut Theorem may be assumed without proof.]

Paper 3, Section II

17F Graph Theory

Let G be a graph of order n. Show that G must contain an independent set of $\left\lceil \sum_{v \in G} \frac{1}{d(v)+1} \right\rceil$ vertices (where $\lceil x \rceil$ denotes the least integer $\geqslant x$).

[*Hint: take a random ordering of the vertices of G, and consider the set of those vertices which are adjacent to no earlier vertex in the ordering.*]

Fix an integer $m < n$ with m dividing n, and suppose that $e(G) = m\binom{n/m}{2}$.

(i) Deduce that G must contain an independent set of m vertices.

(ii) Must G contain an independent set of $m+1$ vertices?

Paper 4, Section II

17F Graph Theory

State Euler's formula relating the number of vertices, edges and faces in a drawing of a connected planar graph. Deduce that every planar graph has chromatic number at most 5.

Show also that any triangle-free planar graph has chromatic number at most 4.

Suppose G is a planar graph which is *minimal* 5-*chromatic;* that is to say, $\chi(G) = 5$ but if H is a subgraph of G with $H \neq G$ then $\chi(H) < 5$. Prove that $\delta(G) \geqslant 5$. Does this remain true if we drop the assumption that G is planar? Justify your answer.

[The Four Colour Theorem may not be assumed.]

Paper 1, Section II

32E Integrable Systems

Define a Poisson structure on an open set $U \subset \mathbb{R}^n$ in terms of an anti-symmetric matrix $\omega^{ab} : U \longrightarrow \mathbb{R}$, where $a, b = 1, \cdots, n$. By considering the Poisson brackets of the coordinate functions x^a show that

$$\sum_{d=1}^{n} \left(\omega^{dc} \frac{\partial \omega^{ab}}{\partial x^d} + \omega^{db} \frac{\partial \omega^{ca}}{\partial x^d} + \omega^{da} \frac{\partial \omega^{bc}}{\partial x^d} \right) = 0 .$$

Now set $n = 3$ and consider $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc} x^c$, where ε^{abc} is the totally antisymmetric symbol on $\mathbb{R}^3$ with $\varepsilon^{123} = 1$. Find a non–constant function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$\{f, x^a\} = 0 , \qquad a = 1, 2, 3 .$$

Consider the Hamiltonian

$$H(x^1, x^2, x^3) = \frac{1}{2} \sum_{a,b=1}^{3} M^{ab} x^a x^b ,$$

where M^{ab} is a constant symmetric matrix and show that the Hamilton equations of motion with $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc} x^c$ are of the form

$$\dot{x}^a = \sum_{b,c=1}^{3} Q^{abc} x^b x^c ,$$

where the constants Q^{abc} should be determined in terms of M^{ab}.

Paper 2, Section II

32E Integrable Systems

Consider the Gelfand–Levitan–Marchenko (GLM) integral equation

$$K(x,y)+F(x+y)+\int_x^{\infty} K(x,z)F(z+y)\,dz \;=\; 0\,,$$

with $F(x) = \sum_1^N \beta_n\, e^{-c_n x}$, where $c_1,\ldots,c_N$ are positive constants and $\beta_1,\ldots,\beta_N$ are constants. Consider separable solutions of the form

$$K(x,y) \;=\; \sum_{n=1}^{N} K_n(x)\, e^{-c_n y}\,,$$

and reduce the GLM equation to a linear system

$$\sum_{m=1}^{N} A_{nm}(x)K_m(x) \;=\; B_n(x)\,,$$

where the matrix $A_{nm}(x)$ and the vector $B_n(x)$ should be determined.

How is K related to solutions of the KdV equation?

Set $N=1$, $c_1=c$, $\beta_1=\beta\exp(8c^3 t)$ where c,β are constants. Show that the corresponding one–soliton solution of the KdV equation is given by

$$u(x,t) \;=\; -\,\frac{4\,\beta_1 c\, e^{-2\,cx}}{(1+(\beta_1/2c)\,e^{-2\,cx})^2}\,.$$

[You may use any facts about the Inverse Scattering Transform without proof.]

Paper 3, Section II

32E Integrable Systems

Consider a vector field

$$V = \alpha\, x \frac{\partial}{\partial x} + \beta\, t \frac{\partial}{\partial t} + \gamma\, v \frac{\partial}{\partial v},$$

on $\mathbb{R}^3$, where α, β and γ are constants. Find the one-parameter group of transformations generated by this vector field.

Find the values of the constants (α, β, γ) such that V generates a Lie point symmetry of the modified KdV equation (mKdV)

$$v_t - 6\, v^2\, v_x + v_{xxx} = 0, \qquad \text{where} \quad v = v(x,t).$$

Show that the function $u = u(x,t)$ given by $u = v^2 + v_x$ satisfies the KdV equation and find a Lie point symmetry of KdV corresponding to the Lie point symmetry of mKdV which you have determined from V.

Paper 1, Section II
22H Linear Analysis

a) State and prove the Banach–Steinhaus Theorem.

[You may use the Baire Category Theorem without proving it.]

b) Let X be a (complex) normed space and $S \subset X$. Prove that if $\{f(x) : x \in S\}$ is a bounded set in $\mathbb{C}$ for every linear functional $f \in X^*$ then there exists $K \geqslant 0$ such that $\|x\| \leqslant K$ for all $x \in S$.

[You may use here the following consequence of the Hahn–Banach Theorem without proving it: for a given $x \in X$, there exists $f \in X^*$ with $\|f\| = 1$ and $|f(x)| = \|x\|$.]

c) Conclude that if two norms $\|.\|_1$ and $\|.\|_2$ on a (complex) vector space V are not equivalent, there exists a linear functional $f : V \to \mathbb{C}$ which is continuous with respect to one of the two norms, and discontinuous with respect to the other.

Paper 2, Section II
22H Linear Analysis

For a sequence $x = (x_1, x_2, \dots)$ with $x_j \in \mathbb{C}$ for all $j \geqslant 1$, let

$$\|x\|_\infty := \sup_{j \geqslant 1} |x_j|$$

and $\ell^\infty = \{x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \geqslant 1 \text{ and } \|x\|_\infty < \infty\}$.

a) Prove that ℓ^∞ is a Banach space.

b) Define

$$c_0 = \{x = (x_1, x_2, \dots) \in \ell^\infty : \lim_{j \to \infty} x_j = 0\}$$

and

$$\ell^1 = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \in \mathbb{N} \text{ and } \|x\|_1 = \sum_{\ell=1}^{\infty} |x_\ell| < \infty \right\}.$$

Show that c_0 is a closed subspace of ℓ^∞. Show that $c_0^* \simeq \ell^1$.

[*Hint: find an isometric isomorphism from* ℓ^1 *to* c_0^*.]

c) Let

$$c_{00} = \{x = (x_1, x_2, \dots) \in \ell^\infty : x_j = 0 \text{ for all } j \text{ large enough}\}.$$

Is c_{00} a closed subspace of ℓ^∞? If not, what is the closure of c_{00}?

Paper 3, Section II

21H Linear Analysis

State and prove the Stone-Weierstrass theorem for real-valued functions.

[You may use without proof the fact that the function $s \to |s|$ can be uniformly approximated by polynomials on $[-1,1]$.]

Paper 4, Section II

22H Linear Analysis

Let X be a Banach space.

a) What does it mean for a bounded linear map $T : X \to X$ to be compact?

b) Let $\mathcal{B}(X)$ be the Banach space of all bounded linear maps $S : X \to X$. Let $\mathcal{B}_0(X)$ be the subset of $\mathcal{B}(X)$ consisting of all compact operators. Show that $\mathcal{B}_0(X)$ is a closed subspace of $\mathcal{B}(X)$. Show that, if $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}_0(X)$, then $ST, TS \in \mathcal{B}_0(X)$.

c) Let

$$X = \ell^2 = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \quad \text{and} \quad \|x\|_2^2 = \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\},$$

and $T : X \to X$ be defined by

$$(Tx)_k = \frac{x_{k+1}}{k+1}.$$

Is T compact? What is the spectrum of T? Explain your answers.

Paper 1, Section II

16G Logic and Set Theory

Show that ${\aleph_\alpha}^2 = \aleph_\alpha$ for all α.

An infinite cardinal m is called *regular* if it cannot be written as a sum of fewer than m cardinals each of which is smaller than m. Prove that $\aleph_0$ and $\aleph_1$ are regular.

Is $\aleph_2$ regular? Is $\aleph_\omega$ regular? Justify your answers.

Paper 2, Section II

16G Logic and Set Theory

Let α be a non-zero ordinal. Prove that there exists a greatest ordinal β such that $\omega^\beta \leqslant \alpha$. Explain why there exists an ordinal γ with $\omega^\beta + \gamma = \alpha$. Prove that γ is unique, and that $\gamma < \alpha$.

A non-zero ordinal α is called *decomposable* if it can be written as the sum of two smaller non-zero ordinals. Deduce that if α is not a power of ω then α is decomposable.

Conversely, prove that if α is a power of ω then α is not decomposable.

[*Hint: consider the cases $\alpha = \omega^\beta$ (β a successor) and $\alpha = \omega^\beta$ (β a limit) separately.*]

Paper 3, Section II

16G Logic and Set Theory

Define the sets V_α, $\alpha \in ON$. What is meant by the *rank* of a set?

Explain briefly why, for every α, there exists a set of rank α.

Let x be a transitive set of rank α. Show that x has an element of rank β for every $\beta < \alpha$.

For which α does there exist a finite set of rank α? For which α does there exist a finite transitive set of rank α? Justify your answers.

[Standard properties of rank may be assumed.]

Paper 4, Section II

16G Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic.

[You do **not** need to give definitions of the various terms involved. You may assume that the set of primitive propositions is countable. You may also assume the Deduction Theorem.]

Explain briefly how your proof should be modified if the set of primitive propositions is allowed to be uncountable.

Paper 1, Section I

6A Mathematical Biology

A delay model for a population N_t consists of

$$N_{t+1} = \frac{rN_t}{1 + b\,N_{t-1}^2} \,,$$

where t is discrete time, $r > 1$ and $b > 0$. Investigate the linear stability about the positive steady state N^*. Show that $r = 2$ is a bifurcation value at which the steady state bifurcates to a periodic solution of period 6.

Paper 2, Section I

6A Mathematical Biology

The population of a certain species subjected to a specific kind of predation is modelled by the difference equation

$$u_{t+1} = a\,\frac{u_t^2}{b^2 + u_t^2} \,, \qquad a > 0 \,.$$

Determine the equilibria and show that if $a^2 > 4\,b^2$ it is possible for the population to be driven to extinction if it becomes less than a critical size which you should find. Explain your reasoning by means of a cobweb diagram.

Paper 3, Section I

6A Mathematical Biology

A population of aerobic bacteria swims in a laterally-infinite layer of fluid occupying $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$, with the top and bottom surfaces in contact with air. Assuming that there is no fluid motion and that all physical quantities depend only on z, the oxygen concentration c and bacterial concentration n obey the coupled equations

$$\frac{\partial c}{\partial t} = D_c \frac{\partial^2 c}{\partial z^2} - kn \, ,$$
$$\frac{\partial n}{\partial t} = D_n \frac{\partial^2 n}{\partial z^2} - \frac{\partial}{\partial z}\left(\mu n \frac{\partial c}{\partial z}\right) \, .$$

Consider first the case in which there is no chemotaxis, so n has the spatially-uniform value $\bar{n}$. Find the steady-state oxygen concentration consistent with the boundary conditions $c\,(\pm d/2) = c_0$. Calculate the Fick's law flux of oxygen into the layer and justify your answer on physical grounds.

Now allowing chemotaxis and cellular diffusion, show that the equilibrium oxygen concentration satisfies

$$\frac{d^2 c}{dz^2} - \frac{kn_0}{D_c} \exp\left(\mu c/D_n\right) = 0 \, ,$$

where n_0 is a suitable normalisation constant that need not be found.

Paper 4, Section I

6A Mathematical Biology

A concentration $u(x,t)$ obeys the differential equation

$$\frac{\partial u}{\partial t} = Du_{xx} + f(u) \, ,$$

in the domain $0 \leqslant x \leqslant L$, with boundary conditions $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = u_0(x)$, and where D is a positive constant. Assume $f(0) = 0$ and $f'(0) > 0$. Linearising the dynamics around $u = 0$, and representing $u(x,t)$ as a suitable Fourier expansion, show that the condition for the linear stability of $u = 0$ can be expressed as the following condition on the domain length

$$L < \pi \left[\frac{D}{f'(0)}\right]^{1/2} \, .$$

Paper 2, Section II

13A Mathematical Biology

The radially symmetric spread of an insect population density $n(r,t)$ in the plane is described by the equation

$$\frac{\partial n}{\partial t} = \frac{D_0}{r}\frac{\partial}{\partial r}\left[r\left(\frac{n}{n_0}\right)^2\frac{\partial n}{\partial r}\right]. \qquad (*)$$

Suppose Q insects are released at $r=0$ at $t=0$. We wish to find a similarity solution to $(*)$ in the form

$$n(r,t) = \frac{n_0}{\lambda^2(t)}\,F\left(\frac{r}{r_0\lambda(t)}\right).$$

Show first that the PDE $(*)$ reduces to an ODE for F if $\lambda(t)$ obeys the equation

$$\lambda^5\,\frac{d\lambda}{dt} = C\,\frac{D_0}{r_0^2},$$

where C is an arbitrary constant (that may be set to unity), and then obtain $\lambda(t)$ and F such that $F(0)=1$ and $F(\xi)=0$ for $\xi \geqslant 1$. Determine r_0 in terms of n_0 and Q. Sketch the function $n(r,t)$ at various times to indicate its qualitative behaviour.

Paper 3, Section II

13A Mathematical Biology

Consider an epidemic model in which $S(x,t)$ is the local population density of susceptibles and $I(x,t)$ is the density of infectives

$$\frac{\partial S}{\partial t} = -\, rIS\ ,$$
$$\frac{\partial I}{\partial t} = D\,\frac{\partial^2 I}{\partial x^2} + rIS - aI\ ,$$

where r, a, and D are positive. If S_0 is a characteristic population value, show that the rescalings $I/S_0 \to I$, $S/S_0 \to S$, $(rS_0/D)^{1/2}\,x \to x$, $rS_0 t \to t$ reduce this system to

$$\frac{\partial S}{\partial t} = -\, IS\ ,$$
$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + IS - \lambda I\ ,$$

where λ should be found.

Travelling wavefront solutions are of the form $S(x,t) = S(z)$, $I(x,t) = I(z)$, where $z = x - ct$ and c is the wave speed, and we seek solutions with boundary conditions $S(\infty) = 1$, $S'(\infty) = 0$, $I(\infty) = I(-\infty) = 0$. Under the travelling-wave assumption reduce the rescaled PDEs to ODEs, and show by linearisation around the leading edge of the advancing front that the requirement that I be non-negative leads to the condition $\lambda < 1$ and hence the wave speed relation

$$c \geqslant 2(1-\lambda)^{1/2}\ , \qquad \lambda < 1\ .$$

Using the two ODEs you have obtained, show that the surviving susceptible population fraction $\sigma = S(-\infty)$ after the passage of the front satisfies

$$\sigma - \lambda \ln \sigma = 1\ ,$$

and sketch σ as a function of λ.

Paper 1, Section II

20G Number Fields

Suppose that m is a square-free positive integer, $m \geqslant 5$, $m \not\equiv 1 \pmod 4$. Show that, if the class number of $K = \mathbb{Q}(\sqrt{-m})$ is prime to 3, then $x^3 = y^2 + m$ has at most two solutions in integers. Assume the m is even.

Paper 2, Section II

20G Number Fields

Calculate the class group of the field $\mathbb{Q}(\sqrt{-14})$.

Paper 4, Section II

20G Number Fields

Suppose that α is a zero of $x^3 - x + 3$ and that $K = \mathbb{Q}(\alpha)$. Show that $[K : \mathbb{Q}] = 3$. Show that O_K, the ring of integers in K, is $O_K = \mathbb{Z}[\alpha]$.

[You may quote any general theorem that you wish, provided that you state it clearly. Note that the discriminant of $x^3 + px + q$ is $-4p^3 - 27q^2$.]

Paper 1, Section I

1G Number Theory

(i) Let N be an integer $\geqslant 2$. Define the addition and multiplication on the set of congruence classes modulo N.

(ii) Let an integer $M \geqslant 1$ have expansion to the base 10 given by $a_s \ldots a_0$. Prove that 11 divides M if and only if $\sum_{i=0}^{s} (-1)^i a_i$ is divisible by 11.

Paper 2, Section I

1G Number Theory

Let p be an odd prime number. If n is an integer prime to p, define $\left(\frac{n}{p}\right)$.

(i) Prove that $\chi(n) = \left(\frac{n}{p}\right)$ defines a homomorphism from $(\mathbb{Z}/p\mathbb{Z})^\times$ to the group $\{\pm 1\}$. What is the value of $\chi(-1)$?

(ii) If $p \equiv 1 \bmod 4$, prove that $\sum_{n=1}^{p-1} \chi(n)\, n = 0$.

Paper 3, Section I

1G Number Theory

(i) Let M and N be positive integers, such that N is not a perfect square. If $M < \sqrt{N}$, show that every solution of the equation

$$x^2 - Ny^2 = M$$

in positive integers x, y comes from some convergent of the continued fraction of $\sqrt{N}$.

(ii) Find a solution in positive integers x, y of

$$x^2 - 29y^2 = 5.$$

Paper 4, Section I

1G Number Theory

Let p be a prime number, and put

$$a_k = kp\,, \quad N_k = a_k^p - 1 \quad (k = 1, 2, \ldots)\,.$$

Prove that a_k has exact order p modulo N_k for all $k \geqslant 1$, and deduce that N_k must be divisible by a prime q with $q \equiv 1 \pmod{p}$. By making a suitable choice of k, prove that there are infinitely many primes q with $q \equiv 1 \pmod{p}$.

Paper 3, Section II

11G Number Theory

State precisely the Miller-Rabin primality test.

(i) Let p be a prime $\geqslant 5$, and define

$$N = \frac{4^p - 1}{3}\,.$$

Prove that N is a composite odd integer, and that N is a pseudo-prime to the base 2.

(ii) Let M be an odd integer greater than 1 such that M is a pseudo-prime to the base 2. Prove that $2^M - 1$ is always a strong pseudo-prime to the base 2.

Paper 4, Section II

11G Number Theory

Let $\mathcal{S}$ be the set of all positive definite binary quadratic forms with integer coefficients. Define the action of the group $SL_2(\mathbb{Z})$ on $\mathcal{S}$, and prove that equivalent forms under this action have the same discriminant.

Find necessary and sufficient conditions for an odd positive integer n, prime to 35, to be properly represented by at least one of the two forms

$$x^2 + xy + 9y^2\,, \qquad 3x^2 + xy + 3y^2\,.$$

Paper 1, Section II

39A Numerical Analysis

(a) State the Householder–John theorem and explain its relation to the convergence analysis of splitting methods for solving a system of linear equations $Ax = b$ with a positive definite matrix A.

(b) Describe the Jacobi method for solving a system $Ax = b$, and deduce from the above theorem that if A is a symmetric positive definite *tridiagonal* matrix,

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ c_1 & a_2 & c_2 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & c_{n-2} & a_{n-1} & c_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix},$$

then the Jacobi method converges.

[*Hint: At the last step, you may find it useful to consider two vectors* $x = (x_1, x_2, \ldots, x_n)$ *and* $y = ((-1)x_1, (-1)^2 x_2, \ldots, (-1)^n x_n)$.]

Paper 2, Section II
39A Numerical Analysis

The inverse discrete Fourier transform $\mathcal{F}_n^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formula

$$\mathbf{x} = \mathcal{F}_n^{-1}\,\mathbf{y}\,, \qquad \text{where} \qquad x_\ell = \sum_{j=0}^{n-1} \omega_n^{j\ell}\, y_j\,, \qquad \ell = 0, \ldots, n-1\,.$$

Here, $\omega_n = \exp(2\pi i/n)$ is the primitive root of unity of degree n, and $n = 2^p$, $p = 1, 2, \ldots$.

(1) Show how to assemble $\mathbf{x} = \mathcal{F}_{2m}^{-1}\,\mathbf{y}$ in a small number of operations if we already know the Fourier transforms of the even and odd portions of $\mathbf{y}$:

$$\mathbf{x}^{(\mathrm{E})} = \mathcal{F}_m^{-1}\,\mathbf{y}^{(\mathrm{E})}\,, \qquad \mathbf{x}^{(\mathrm{O})} = \mathcal{F}_m^{-1}\,\mathbf{y}^{(\mathrm{O})}\,.$$

(2) Describe the Fast Fourier Transform (FFT) method for evaluating $\mathbf{x}$ and draw a relevant diagram for $n = 8$.

(3) Find the costs of the FFT for $n = 2^p$ (only multiplications count).

(4) For $n = 4$, using the FFT technique, find

$$\mathbf{x} = \mathcal{F}_4^{-1}\,\mathbf{y}\,, \qquad \text{for} \qquad \mathbf{y} = [1, 1, -1, -1]\,, \qquad \text{and} \qquad \mathbf{y} = [1, -1, 1, -1]\,.$$

Paper 3, Section II
39A Numerical Analysis

The Poisson equation $\nabla^2 u = f$ in the unit square $\Omega = [0,1] \times [0,1]$, $u = 0$ on $\partial\Omega$, is discretized with the five-point formula

$$u_{i,j-1} + u_{i,j+1} + u_{i+1,j} + u_{i-1,j} - 4\,u_{i,j} = h^2 f_{i,j}\,,$$

where $1 \leqslant i,\ j \leqslant M$, $u_{i,j} \approx u(ih, jh)$ and (ih, jh) are grid points.

Let $u(x,y)$ be the exact solution, and let $e_{i,j} = u_{i,j} - u(ih, jh)$ be the error of the five-point formula at the (i,j)th grid point. Justifying each step, prove that

$$\|\mathbf{e}\| = \left[\sum_{i,j=1}^{M} |\,e_{i,j}|^2\right]^{1/2} \leqslant ch \quad \text{for sufficiently small} \quad h > 0\,,$$

where c is some constant independent of h.

Paper 4, Section II

39A Numerical Analysis

An s-stage explicit Runge–Kutta method of order p, with constant step size $h > 0$, is applied to the differential equation $y' = \lambda y, \ t \geqslant 0$.

(a) Prove that

$$y_{n+1} = P_s(\lambda h)\, y_n \, .$$

where P_s is a polynomial of degree s.

(b) Prove that the order p of any s-stage explicit Runge–Kutta method satisfies the inequality $p \leqslant s$ and, for $p = s$, write down an explicit expression for P_s.

(c) Prove that no explicit Runge–Kutta method can be A-stable.

Paper 2, Section II

29J Optimization and Control

(a) Suppose that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} V_{XX} & V_{XY} \\ V_{YX} & V_{YY} \end{pmatrix} \right).$$

Prove that conditional on $Y = y$, the distribution of X is again multivariate normal, with mean $\mu_X + V_{XY} V_{YY}^{-1}(y - \mu_Y)$ and covariance $V_{XX} - V_{XY} V_{YY}^{-1} V_{YX}$.

(b) The $\mathbb{R}^d$-valued process X evolves in discrete time according to the dynamics

$$X_{t+1} = AX_t + \varepsilon_{t+1},$$

where A is a constant $d \times d$ matrix, and ε_t are independent, with common $N(0, \Sigma_\varepsilon)$ distribution. The process X is not observed directly; instead, all that is seen is the process Y defined as

$$Y_t = CX_t + \eta_t,$$

where η_t are independent of each other and of the ε_t, with common $N(0, \Sigma_\eta)$ distribution.

If the observer has the prior distribution $X_0 \sim N(\hat{X}_0, V_0)$ for X_0, prove that at all later times the distribution of X_t conditional on $\mathcal{Y}_t \equiv (Y_1, \ldots, Y_t)$ is again normally distributed, with mean $\hat{X}_t$ and covariance V_t which evolve as

$$\begin{aligned} \hat{X}_{t+1} &= A\hat{X}_t + M_t C^T (\Sigma_\eta + C M_t C^T)^{-1} (Y_{t+1} - CA\hat{X}_t), \\ V_{t+1} &= M_t - M_t C^T (\Sigma_\eta + C M_t C^T)^{-1} C M_t, \end{aligned}$$

where

$$M_t = A V_t A^T + \Sigma_\varepsilon.$$

(c) In the special case where both X and Y are one-dimensional, and $A = C = 1$, $\Sigma_\varepsilon = 0$, find the form of the updating recursion. Show in particular that

$$\frac{1}{V_{t+1}} = \frac{1}{V_t} + \frac{1}{\Sigma_\eta}$$

and that

$$\frac{\hat{X}_{t+1}}{V_{t+1}} = \frac{\hat{X}_t}{V_t} + \frac{Y_{t+1}}{\Sigma_\eta}.$$

Hence deduce that, with probability one,

$$\lim_{t \to \infty} \hat{X}_t = \lim_{t \to \infty} t^{-1} \sum_{j=1}^{t} Y_j.$$

Paper 3, Section II

28J Optimization and Control

Consider an infinite-horizon controlled Markov process having per-period costs $c(x,u) \geqslant 0$, where $x \in \mathcal{X}$ is the state of the system, and $u \in \mathcal{U}$ is the control. Costs are discounted at rate $\beta \in (0,1]$, so that the objective to be minimized is

$$\mathbb{E}\left[\sum_{t \geqslant 0} \beta^t c(X_t, u_t) \mid X_0 = x \right].$$

What is meant by a *policy* π for this problem?

Let $\mathcal{L}$ denote the dynamic programming operator

$$\mathcal{L}f(x) \equiv \inf_{u \in \mathcal{U}} \left\{ c(x,u) + \beta \mathbb{E}\left[f(X_1) \,\middle|\, X_0 = x, u_0 = u \right] \right\}.$$

Further, let F denote the value of the optimal control problem:

$$F(x) = \inf_{\pi} \mathbb{E}^{\pi}\left[\sum_{t \geqslant 0} \beta^t c(X_t, u_t) \mid X_0 = x \right],$$

where the infimum is taken over all policies π, and $\mathbb{E}^{\pi}$ denotes expectation under policy π. Show that the functions F_t defined by

$$F_{t+1} = \mathcal{L}F_t \quad (t \geqslant 0), \qquad F_0 \equiv 0$$

increase to a limit $F_\infty \in [0, \infty]$. Prove that $F_\infty \leqslant F$. Prove that $F = \mathcal{L}F$.

Suppose that $\Phi = \mathcal{L}\Phi \geqslant 0$. Prove that $\Phi \geqslant F$.

[You may assume that there is a function $u_* : \mathcal{X} \to \mathcal{U}$ such that

$$\mathcal{L}\Phi(x) = c(x, u_*(x)) + \beta \mathbb{E}\left[\Phi(X_1) \,\middle|\, X_0 = x, u_0 = u_*(x) \right],$$

though the result remains true without this simplifying assumption.]

Paper 4, Section II

28J Optimization and Control

Dr Seuss' wealth x_t at time t evolves as

$$\frac{dx}{dt} = rx_t + \ell_t - c_t,$$

where $r > 0$ is the rate of interest earned, ℓ_t is his intensity of working ($0 \leqslant \ell \leqslant 1$), and c_t is his rate of consumption. His initial wealth $x_0 > 0$ is given, and his objective is to maximize

$$\int_0^T U(c_t, \ell_t)\, dt,$$

where $U(c, \ell) = c^\alpha (1-\ell)^\beta$, and T is the (fixed) time his contract expires. The constants α and β satisfy the inequalities $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta > 1$. At all times, c_t must be non-negative, and his final wealth x_T must be non-negative. Establish the following properties of the optimal solution (x^*, c^*, ℓ^*):

(i) $\beta c_t^* = \alpha(1 - \ell_t^*)$;

(ii) $c_t^* \propto e^{-\gamma r t}$, where $\gamma \equiv (\beta - 1 + \alpha)^{-1}$;

(iii) $x_t^* = Ae^{rt} + Be^{-\gamma r t} - r^{-1}$ for some constants A and B.

Hence deduce that the optimal wealth is

$$x_t^* = \frac{(1 - e^{-\gamma r T}(1 + rx_0))e^{rt} + ((1 + rx_0)e^{rT} - 1)e^{-\gamma r t}}{r(e^{rT} - e^{-\gamma r T})} - \frac{1}{r}.$$

Paper 1, Section II

30E Partial Differential Equations

(a) Solve by using the method of characteristics

$$x_1 \frac{\partial}{\partial x_1} u + 2 x_2 \frac{\partial}{\partial x_2} u = 5 u , \qquad u(x_1, 1) = g(x_1) ,$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. What is the maximal domain in $\mathbb{R}^2$ in which u is a solution of the Cauchy problem?

(b) Prove that the function

$$u(x,t) = \begin{cases} 0 , & x < 0 , \, t > 0 , \\ x/t , & 0 < x < t , \, t > 0 , \\ 1 , & x > t > 0 , \end{cases}$$

is a weak solution of the Burgers equation

$$\frac{\partial}{\partial t} u + \frac{1}{2} \frac{\partial}{\partial x} u^2 = 0 , \qquad x \in \mathbb{R}, \, t > 0 , \tag{$*$}$$

with initial data

$$u(x,0) = \begin{cases} 0 , & x < 0 , \\ 1 , & x > 0 . \end{cases}$$

(c) Let $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$ be a piecewise C^1-function with a jump discontinuity along the curve

$$\Gamma : x = s(t)$$

and let u solve the Burgers equation $(*)$ on both sides of Γ. Prove that u is a weak solution of (1) if and only if

$$\dot{s}(t) = \frac{1}{2} \left(u_l(t) + u_r(t) \right)$$

holds, where $u_l(t)$, $u_r(t)$ are the one-sided limits

$$u_l(t) = \lim_{x \nearrow s(t)^-} u(x,t) , \qquad u_r(t) = \lim_{x \searrow s(t)^+} u(x,t) .$$

[*Hint: Multiply the equation by a test function $\phi \in C_0^\infty(\mathbb{R} \times [0, \infty))$, split the integral appropriately and integrate by parts. Consider how the unit normal vector along Γ can be expressed in terms of $\dot{s}$.*]

Paper 2, Section II

31E Partial Differential Equations

(a) State the Lax-Milgram lemma. Use it to prove that there exists a unique function u in the space

$$H^2_\partial(\Omega) = \left\{u \in H^2(\Omega)\,; u|_{\partial\Omega} = \partial u/\partial\gamma|_{\partial\Omega} = 0\right\} ,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and γ its outwards unit normal vector, which is the weak solution of the equations

$$\begin{aligned}\Delta^2 u &= f \quad \text{in} \quad \Omega\,,\\ u = \frac{\partial u}{\partial\gamma} &= 0 \quad \text{on} \quad \partial\,\Omega\,,\end{aligned}$$

for $f \in L^2(\Omega)$, Δ the Laplacian and $\Delta^2 = \Delta\Delta$.

[*Hint: Use regularity of the solution of the Dirichlet problem for the Poisson equation.*]

(b) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $u \in H^1(\Omega)$ and denote

$$\bar{u} = \int_\Omega u\, d^n x \Big/ \int_\Omega d^n x\ .$$

The following Poincaré-type inequality is known to hold

$$\|u - \bar{u}\|_{L^2} \leqslant C\|\nabla u\|_{L^2}\,,$$

where C only depends on Ω. Use the Lax-Milgram lemma and this Poincaré-type inequality to prove that the Neumann problem

$$\begin{aligned}\Delta u &= f \quad \text{in} \quad \Omega\,,\\ \frac{\partial u}{\partial\gamma} &= 0 \quad \text{on} \quad \partial\,\Omega\,,\end{aligned}$$

has a unique weak solution in the space

$$H^1_-(\Omega) = H^1(\Omega) \cap \{u : \Omega \to \mathbb{R}\,; \bar{u} = 0\}$$

if and only if $\bar{f} = 0$.

Paper 3, Section II

30E Partial Differential Equations

Consider the Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n\,,\ t > 0\,,$$

for complex-valued solutions $\Psi(x,t)$ and where Δ is the Laplacian.

(a) Derive, by using a Fourier transform and its inversion, the fundamental solution of the Schrödinger equation. Obtain the solution of the initial value problem

$$\begin{aligned} i\partial_t \Psi &= -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n,\quad t > 0\ , \\ \Psi(x,0) &= f(x)\,, \qquad x \in \mathbb{R}^n\,, \end{aligned}$$

as a convolution.

(b) Consider the Wigner-transform of the solution of the Schrödinger equation

$$w(x,\xi,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \Psi(x + \tfrac{1}{2}\,y,\ t)\ \bar{\Psi}(x - \tfrac{1}{2}\,y,\ t)\ e^{-iy\cdot\xi}\ \mathrm{d}^n y\,,$$

defined for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $t > 0$. Derive an evolution equation for w by using the Schrödinger equation. Write down the solution of this evolution equation for given initial data $w(x,\xi,0) = g(x,\xi)$.

Paper 4, Section II

30E Partial Differential Equations

a) Solve the Dirichlet problem for the Laplace equation in a disc in $\mathbb{R}^2$

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad G = \{x^2 + y^2 < R^2\} \subseteq \mathbb{R}^2\ ,\ R > 0\,, \\ u = u_D \quad &\text{on} \quad \partial G\,, \end{aligned}$$

using polar coordinates (r,φ) and separation of variables, $u(x,y) = R(r)\Theta(\varphi)$. Then use the ansatz $R(r) = r^\alpha$ for the radial function.

b) Solve the Dirichlet problem for the Laplace equation in a square in $\mathbb{R}^2$

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad G = [0,a]\times[0,a]\,, \\ &u(x,0) = f_1(x)\,, \qquad u(x,a) = f_2(x)\,, \qquad u(0,y) = f_3(y)\,, \qquad u(a,y) = f_4(y)\,. \end{aligned}$$

Paper 1, Section II

33C Principles of Quantum Mechanics

Two states $|j_1\ m_1\rangle_1$, $|j_2\ m_2\rangle_2$, with angular momenta j_1, j_2, are combined to form states $|J\ M\rangle$ with total angular momentum

$$J = |j_1 - j_2|,\ |j_1 - j_2| + 1\,,\ \ldots\,,\ j_1 + j_2\,.$$

Write down the state with $J = M = j_1 + j_2$ in terms of the original angular momentum states. Briefly describe how the other combined angular momentum states may be found in terms of the original angular momentum states.

If $j_1 = j_2 = j$, explain why the state with $J = 0$ must be of the form

$$|0\ 0\rangle = \sum_{m=-j}^{j} \alpha_m |j\ m\rangle_1 |j\ {-m}\rangle_2\,.$$

By considering $J_+|0\ 0\rangle$, determine a relation between α_{m+1} and α_m, hence find α_m.

If the system is in the state $|j\ j\rangle_1|j\ {-j}\rangle_2$ what is the probability, written in terms of j, of measuring the combined total angular momentum to be zero?

[Standard angular momentum states $|j\ m\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_{\pm}|j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m{\pm}1\rangle\,.$$

Units in which $\hbar = 1$ have been used throughout.]

Paper 2, Section II

33C Principles of Quantum Mechanics

Consider a joint eigenstate of $\mathbf{J}^2$ and J_3, $|j\ m\rangle$. Write down a unitary operator $U(\mathbf{n},\theta)$ for rotation of the state by an angle θ about an axis with direction $\mathbf{n}$, where $\mathbf{n}$ is a unit vector. How would a state with zero orbital angular momentum transform under such a rotation?

What is the relation between the angular momentum operator $\mathbf{J}$ and the Pauli matrices $\boldsymbol{\sigma}$ when $j=\frac{1}{2}$? Explicitly calculate $(\mathbf{J}\cdot\mathbf{a})^2$, for an arbitrary real vector $\mathbf{a}$, in this case. What are the eigenvalues of the operator $\mathbf{J}\cdot\mathbf{a}$? Show that the unitary rotation operator for $j=\frac{1}{2}$ can be expressed as

$$U(\mathbf{n},\theta) = \cos\frac{\theta}{2} - i\,\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}\,. \qquad (*)$$

Starting with a state $|\frac{1}{2}\ m\rangle$ the component of angular momentum along a direction $\mathbf{n}'$, making and angle θ with the z-axis, is susequently measured to be m'. Immediately after this measurement the state is $|\frac{1}{2}\ m'\rangle_\theta$. Write down an eigenvalue equation for $|\frac{1}{2}\ m'\rangle_\theta$ in terms of $\mathbf{n}'\cdot\mathbf{J}$. Show that the probability for measuring an angular momentum of $m'\hbar$ along the direction $\mathbf{n}'$ is, assuming $\mathbf{n}'$ is in the x-z plane,

$$\left|\langle\tfrac{1}{2}\ m|\tfrac{1}{2}\ m'\rangle_\theta\right|^2 = \left|\langle\tfrac{1}{2}\ m|U(\mathbf{y},\theta)|\tfrac{1}{2}\ m'\rangle\right|^2,$$

where $\mathbf{y}$ is a unit vector in the y-direction. Using $(*)$ show that the probability that $m=+\frac{1}{2}$, $m'=-\frac{1}{2}$ is of the form

$$A+B\cos^2\frac{\theta}{2}\,,$$

determining the integers A and B in the process.

[Assume $\hbar=1$. The Pauli matrices are

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

]

Paper 3, Section II

33C Principles of Quantum Mechanics

What are the commutation relations between the position operator $\hat{x}$ and momentum operator $\hat{p}$? Show that this is consistent with $\hat{x}$, $\hat{p}$ being hermitian.

The annihilation operator for a harmonic oscillator is

$$a = \sqrt{\frac{1}{2\hbar}}\,(\hat{x} + i\hat{p})$$

in units where the mass and frequency of the oscillator are 1. Derive the relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian

$$H = \tfrac{1}{2}\,\hat{p}^2 + \tfrac{1}{2}\,\hat{x}^2$$

in terms of the operator $N = a^\dagger a$.

Assume there exists a unique ground state $|0\rangle$ of H such that $a|0\rangle = 0$. Explain how the space of eigenstates $|n\rangle$, is formed, and deduce the energy eigenvalues for these states. Show that

$$a|n\rangle = A|n{-}1\rangle\,, \qquad a^\dagger|n\rangle = B|n{+}1\rangle\,,$$

finding A and B in terms of n.

Calculate the energy eigenvalues of the Hamiltonian for two harmonic oscillators

$$H = H_1 + H_2\,, \qquad H_i = \tfrac{1}{2}\,\hat{p}_i^2 + \tfrac{1}{2}\,\hat{x}_i^2\,, \quad i = 1, 2\,.$$

What is the degeneracy of the n^{th} energy level? Suppose that the two oscillators are then coupled by adding the extra term

$$\Delta H = \lambda \hat{x}_1\,\hat{x}_2$$

to H, where $\lambda \ll 1$. Calculate the energies for the states of the unperturbed system with the three lowest energy eigenvalues to first order in λ using perturbation theory.

[You may assume standard perturbation theory results.]

Paper 4, Section II

32C Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + V(t)\,,$$

where H_0 is independent of time. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|a\rangle$ and $|b\rangle$ be orthonormal eigenstates of H_0 with eigenvalues E_a and E_b respectively. Assume $V(t) = 0$ for $t \leqslant 0$. Show that if the system is initially, at $t = 0$, in the state $|a\rangle$ then the probability of measuring it to be the state $|b\rangle$ after a time t is

$$\frac{1}{\hbar^2}\left|\int_0^t dt' \langle b|V(t')|a\rangle e^{\,i(E_b-E_a)t'/\hbar}\right|^2 \qquad (*)$$

to order $V(t)^2$.

Suppose a system has a basis of just two orthonormal states $|1\rangle$ and $|2\rangle$, with respect to which

$$H_0 \;=\; E\,I\,, \qquad V(t) = vt\,\sigma_1\,, \quad t \geqslant 0\,,$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Use $(*)$ to calculate the probability of a transition from state $|1\rangle$ to state $|2\rangle$ after a time t to order v^2.

Show that the time dependent Schrödinger equation has a solution

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\Big(Et\,I + \tfrac{1}{2}vt^2\,\sigma_1\Big)\right)|\psi(0)\rangle\,.$$

Calculate the transition probability exactly. Hence find the condition for the order v^2 approximation to be valid.

Paper 1, Section II

28J Principles of Statistics

The distribution of a random variable X is obtained from the binomial distribution $\mathcal{B}(n;\Pi)$ by conditioning on $X > 0$; here $\Pi \in (0,1)$ is an unknown probability parameter and n is known. Show that the distributions of X form an exponential family and identify the natural sufficient statistic T, natural parameter Φ, and cumulant function $k(\phi)$. Using general properties of the cumulant function, compute the mean and variance of X when $\Pi = \pi$. Write down an equation for the maximum likelihood estimate $\widehat{\Pi}$ of Π and explain why, when $\Pi = \pi$, the distribution of $\widehat{\Pi}$ is approximately normal $\mathcal{N}(\pi, \pi(1-\pi)/n)$ for large n.

Suppose we observe $X = 1$. It is suggested that, since the condition $X > 0$ is then automatically satisfied, general principles of inference require that the inference to be drawn should be the same as if the distribution of X had been $\mathcal{B}(n;\Pi)$ and we had observed $X = 1$. Comment briefly on this suggestion.

Paper 2, Section II

28J Principles of Statistics

Define the *Kolmogorov–Smirnov* statistic for testing the null hypothesis that real random variables $X_1, \ldots, X_n$ are independently and identically distributed with specified continuous, strictly increasing distribution function F, and show that its null distribution does not depend on F.

A composite hypothesis H_0 specifies that, when the unknown positive parameter Θ takes value θ, the random variables $X_1, \ldots, X_n$ arise independently from the uniform distribution $\mathrm{U}[0,\theta]$. Letting $J := \arg\max_{1 \leqslant i \leqslant n} X_i$, show that, under H_0, the statistic (J, X_J) is sufficient for Θ. Show further that, given $\{J = j,\ X_j = \xi\}$, the random variables $(X_i : i \neq j)$ are independent and have the $\mathrm{U}[0,\xi]$ distribution. How might you apply the Kolmogorov–Smirnov test to test the hypothesis H_0?

Paper 3, Section II

27J Principles of Statistics

Define the *normal* and *extensive* form solutions of a Bayesian statistical decision problem involving parameter Θ, random variable X, and loss function $L(\theta, a)$. How are they related? Let $R_0 = R_0(\Pi)$ be the Bayes loss of the optimal act when $\Theta \sim \Pi$ and no data can be observed. Express the Bayes risk R_1 of the optimal statistical decision rule in terms of R_0 and the joint distribution of (Θ, X).

The real parameter Θ has distribution Π, having probability density function $\pi(\cdot)$. Consider the problem of specifying a set $S \subseteq \mathbb{R}$ such that the loss when $\Theta = \theta$ is $L(\theta, S) = c\,|S| - \mathbf{1}_S(\theta)$, where $\mathbf{1}_S$ is the indicator function of S, where $c > 0$, and where $|S| = \int_S dx$. Show that the "highest density" region $S^* := \{\theta : \pi(\theta) \geqslant c\}$ supplies a Bayes act for this decision problem, and explain why $R_0(\Pi) \leqslant 0$.

For the case $\Theta \sim \mathcal{N}(\mu, \sigma^2)$, find an expression for R_0 in terms of the standard normal distribution function Φ.

Suppose now that $c = 0.5$, that $\Theta \sim \mathcal{N}(0, 1)$ and that $X|\Theta \sim \mathcal{N}(\Theta, 1/9)$. Show that $R_1 < R_0$.

Paper 4, Section II

27J Principles of Statistics

Define *completeness* and *bounded completeness* of a statistic T in a statistical experiment.

Random variables X_1, X_2, X_3 are generated as $X_i = \Theta^{1/2}\, Z + (1 - \Theta)^{1/2}\, Y_i$, where Z, Y_1, Y_2, Y_3 are independently standard normal $\mathcal{N}(0, 1)$, and the parameter Θ takes values in $(0, 1)$. What is the joint distribution of (X_1, X_2, X_3) when $\Theta = \theta$? Write down its density function, and show that a minimal sufficient statistic for Θ based on (X_1, X_2, X_3) is $T = (T_1, T_2) := (\sum_{i=1}^3 X_i^2, (\sum_{i=1}^3 X_i)^2)$.

[*Hint: You may use that if I is the $n \times n$ identity matrix and J is the $n \times n$ matrix all of whose entries are 1, then $aI + bJ$ has determinant $a^{n-1}(a + nb)$, and inverse $cI + dJ$ with $c = 1/a$, $d = -b/(a(a + nb))$.*]

What is $\mathbb{E}_\theta(T_1)$? Is T complete for Θ?

Let $S := \mathrm{Prob}(X_1^2 \leqslant 1 \mid T)$. Show that $\mathbb{E}_\theta(S)$ is a positive constant c which does not depend on θ, but that S is not identically equal to c. Is T boundedly complete for Θ?

Paper 1, Section II

26I Probability and Measure

State Carathéodory's extension theorem. Define all terms used in the statement.

Let $\mathcal{A}$ be the ring of finite unions of disjoint bounded intervals of the form

$$A = \bigcup_{i=1}^{m} (a_i, b_i]$$

where $m \in \mathbb{Z}^+$ and $a_1 < b_1 < \ldots < a_m < b_m$. Consider the set function μ defined on $\mathcal{A}$ by

$$\mu(A) = \sum_{i=1}^{m} (b_i - a_i) .$$

You may assume that μ is additive. Show that for any decreasing sequence $(B_n : n \in \mathbb{N})$ in $\mathcal{A}$ with empty intersection we have $\mu(B_n) \to 0$ as $n \to \infty$.

Explain how this fact can be used in conjunction with Carathéodory's extension theorem to prove the existence of Lebesgue measure.

Paper 2, Section II

26I Probability and Measure

Show that any two probability measures which agree on a π-system also agree on the σ-algebra generated by that π-system.

State Fubini's theorem for non-negative measurable functions.

Let μ denote Lebesgue measure on $\mathbb{R}^2$. Fix $s \in [0, 1)$. Set $c = \sqrt{1 - s^2}$ and $\lambda = \sqrt{c}$. Consider the linear maps $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (\lambda^{-1}x,\, \lambda y) , \qquad g(x, y) = (x,\, sx + y) , \qquad h(x, y) = (x - sy,\, y) .$$

Show that $\mu = \mu \circ f^{-1}$ and that $\mu = \mu \circ g^{-1}$. You must justify any assertion you make concerning the values taken by μ.

Compute $r = f \circ h \circ g \circ f$. Deduce that μ is invariant under rotations.

Paper 3, Section II

25I Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables with common density function

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Fix $\alpha \in [0,1]$ and set

$$Y_n = \operatorname{sgn}(X_n)|X_n|^\alpha, \quad S_n = Y_1 + \ldots + Y_n.$$

Show that for all $\alpha \in [0,1]$ the sequence of random variables S_n/n converges in distribution and determine the limit.

[*Hint: In the case* $\alpha = 1$ *it may be useful to prove that* $\mathbb{E}(e^{iuX_1}) = e^{-|u|}$, *for all* $u \in \mathbb{R}$.]

Show further that for all $\alpha \in [0,1/2)$ the sequence of random variables $S_n/\sqrt{n}$ converges in distribution and determine the limit.

[You should state clearly any result about random variables from the course to which you appeal. You are not expected to evaluate explicitly the integral

$$m(\alpha) = \int_0^\infty \frac{x^\alpha}{\pi(1+x^2)} dx. \quad]$$

Paper 4, Section II

25I Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent normal random variables having mean 0 and variance 1. Set $S_n = X_1 + \ldots + X_n$ and $U_n = S_n - \lfloor S_n \rfloor$. Thus U_n is the fractional part of S_n. Show that U_n converges to U in distribution, as $n \to \infty$ where U is uniformly distributed on $[0,1]$.

Paper 1, Section II

19F Representation Theory

(i) Let N be a normal subgroup of the finite group G. Without giving detailed proofs, define the process of lifting characters from G/N to G. State also the orthogonality relations for G.

(ii) Let a, b be the following two permutations in S_{12},

$$a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)\,,$$

$$b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)\,,$$

and let $G = \langle a, b\rangle$, a subgroup of S_{12}. Prove that G is a group of order 12 and list the conjugacy classes of G. By identifying a normal subgroup of G of index 4 and lifting irreducible characters, calculate all the linear characters of G. Calculate the complete character table of G. By considering 6th roots of unity, find explicit matrix representations affording the non-linear characters of G.

Paper 2, Section II

19F Representation Theory

Define the concepts of induction and restriction of characters. State and prove the Frobenius Reciprocity Theorem.

Let H be a subgroup of G and let $g \in G$. We write $\mathcal{C}(g)$ for the conjugacy class of g in G, and write $C_G(g)$ for the centraliser of g in G. Suppose that $H \cap \mathcal{C}(g)$ breaks up into m conjugacy classes of H, with representatives $x_1, x_2, \ldots, x_m$.

Let ψ be a character of H. Writing $\mathrm{Ind}_H^G(\psi)$ for the induced character, prove that

(i) if no element of $\mathcal{C}(g)$ lies in H, then $\mathrm{Ind}_H^G(\psi)(g) = 0$,

(ii) if some element of $\mathcal{C}(g)$ lies in H, then

$$\mathrm{Ind}_H^G(\psi)(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|}\,.$$

Let $G = S_4$ and let $H = \langle a, b\rangle$, where $a = (1\ 2\ 3\ 4)$ and $b = (1\ 3)$. Identify H as a dihedral group and write down its character table. Restrict each G-conjugacy class to H and calculate the H-conjugacy classes contained in each restriction. Given a character ψ of H, express $\mathrm{Ind}_H^G(\psi)(g)$ in terms of ψ, where g runs through a set of conjugacy classes of G. Use your calculation to find the values of all the irreducible characters of H induced to G.

Paper 3, Section II

19F Representation Theory

Show that the degree of a complex irreducible character of a finite group is a factor of the order of the group.

State and prove Burnside's $p^a q^b$ theorem. You should quote clearly any results you use.

Prove that for any group of odd order n having precisely k conjugacy classes, the integer $n - k$ is divisible by 16.

Paper 4, Section II

19F Representation Theory

Define the circle group $U(1)$. Give a complete list of the irreducible representations of $U(1)$.

Define the spin group $G = SU(2)$, and explain briefly why it is homeomorphic to the unit 3-sphere in $\mathbb{R}^4$. Identify the conjugacy classes of G and describe the classification of the irreducible representations of G. Identify the characters afforded by the irreducible representations. You need not give detailed proofs but you should define all the terms you use.

Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices by conjugation, where $A \in SU(2)$ acts by

$$A : M \mapsto A_1 M A_1{}^{-1},$$

in which A_1 denotes the 3×3 block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Show that this gives a representation of G and decompose it into irreducibles.

Paper 1, Section II

23G Riemann Surfaces

Given a lattice $\Lambda \subset \mathbb{C}$, we may define the corresponding Weierstrass $\wp$-function to be the unique even Λ-periodic elliptic function $\wp$ with poles only on Λ and for which $\wp(z) - 1/z^2 \to 0$ as $z \to 0$. For $w \notin \Lambda$, we set

$$f(z) = \det \begin{pmatrix} 1 & 1 & 1 \\ \wp(z) & \wp(w) & \wp(-z-w) \\ \wp'(z) & \wp'(w) & \wp'(-z-w) \end{pmatrix},$$

an elliptic function with periods Λ. By considering the poles of f, show that f has valency at most 4 (i.e. is at most 4 to 1 on a period parallelogram).

If $w \notin \frac{1}{3}\Lambda$, show that f has at least six distinct zeros. If $w \in \frac{1}{3}\Lambda$, show that f has at least four distinct zeros, at least one of which is a multiple zero. Deduce that the meromorphic function f is identically zero.

If z_1, z_2, z_3 are distinct non-lattice points in a period parallelogram such that $z_1 + z_2 + z_3 \in \Lambda$, what can be said about the points $(\wp(z_i), \wp'(z_i)) \in \mathbb{C}^2$ $(i = 1, 2, 3)$?

Paper 2, Section II

23G Riemann Surfaces

Given a complete analytic function $\mathcal{F}$ on a domain $U \subset \mathbb{C}$, describe *briefly* how the space of germs construction yields a Riemann surface R associated to $\mathcal{F}$ together with a covering map $\pi : R \to U$ (*proofs not required*).

In the case when π is regular, explain briefly how, given a point $P \in U$, any closed curve in U with initial and final points P yields a permutation of the set $\pi^{-1}(P)$.

Now consider the Riemann surface R associated with the complete analytic function

$$(z^2 - 1)^{1/2} + (z^2 - 4)^{1/2}$$

on $U = \mathbb{C} \setminus \{\pm 1, \pm 2\}$, with regular covering map $\pi : R \to U$. Which subgroup of the full symmetric group of $\pi^{-1}(P)$ is obtained in this way from all such closed curves (with initial and final points P)?

Paper 3, Section II

22G Riemann Surfaces

Show that the analytic isomorphisms (i.e. conformal equivalences) of the Riemann sphere $\mathbb{C}_\infty$ to itself are given by the non-constant Möbius transformations.

State the Riemann–Hurwitz formula for a non-constant analytic map between compact Riemann surfaces, carefully explaining the terms which occur.

Suppose now that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an analytic map of degree 2; show that there exist Möbius transformations S and T such that

$$SfT : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is the map given by $z \mapsto z^2$.

Paper 1, Section I

5J Statistical Modelling

Consider a binomial generalised linear model for data $y_1, ..., y_n$ modelled as realisations of independent $Y_i \sim \text{Bin}(1, \mu_i)$ and logit link $\mu_i = e^{\beta x_i}/(1+e^{\beta x_i})$ for some known constants x_i, $i = 1, \ldots, n$, and unknown scalar parameter β. Find the log-likelihood for β, and the likelihood equation that must be solved to find the maximum likelihood estimator $\hat{\beta}$ of β. Compute the second derivative of the log-likelihood for β, and explain the algorithm you would use to find $\hat{\beta}$.

Paper 2, Section I

5J Statistical Modelling

Suppose you have a parametric model consisting of probability mass functions $f(y;\theta)$, $\theta \in \Theta \subset \mathbb{R}$. Given a sample $Y_1, ..., Y_n$ from $f(y;\theta)$, define the maximum likelihood estimator $\hat{\theta}_n$ for θ and, assuming standard regularity conditions hold, state the asymptotic distribution of $\sqrt{n}\,(\hat{\theta}_n - \theta)$.

Compute the Fisher information of a single observation in the case where $f(y;\theta)$ is the probability mass function of a Poisson random variable with parameter θ. If $Y_1, ..., Y_n$ are independent and identically distributed random variables having a Poisson distribution with parameter θ, show that $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $S = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ are unbiased estimators for θ. Without calculating the variance of S, show that there is no reason to prefer S over $\overline{Y}$.

[You may use the fact that the asymptotic variance of $\sqrt{n}\,(\hat{\theta}_n - \theta)$ is a lower bound for the variance of any unbiased estimator.]

Paper 3, Section I

5J Statistical Modelling

Consider the linear model $Y = X\beta + \varepsilon$, where Y is a $n \times 1$ random vector, $\varepsilon \sim N_n(0, \sigma^2 I)$, and where the $n \times p$ nonrandom matrix X is known and has full column rank p. Derive the maximum likelihood estimator $\hat{\sigma}^2$ of σ^2. Without using Cochran's theorem, show carefully that $\hat{\sigma}^2$ is biased. Suggest another estimator $\tilde{\sigma}^2$ for σ^2 that is unbiased.

Paper 4, Section I

5J Statistical Modelling

Below is a simplified 1993 dataset of US cars. The columns list index, make, model, price (in \$1000), miles per gallon, number of passengers, length and width in inches, and weight (in pounds). The data are displayed in R as follows (abbreviated):

```
> cars
       make    model price mpg psngr length width weight
1     Acura  Integra  15.9  31     5    177    68   2705
2     Acura   Legend  33.9  25     5    195    71   3560
3      Audi       90  29.1  26     5    180    67   3375
4      Audi      100  37.7  26     6    193    70   3405
5       BMW     535i  30.0  30     4    186    69   3640
        ...               ...                       ...
92    Volvo      240  22.7  28     5    190    67   2985
93    Volvo      850  26.7  28     5    184    69   3245
```

It is reasonable to assume that prices for different makes of car are independent. We model the logarithm of the price as a linear combination of the other quantitative properties of the cars and an error term. Write down this model mathematically. How would you instruct R to fit this model and assign it to a variable "fit"?

R provides the following (slightly abbreviated) summary:

```
> summary(fit)

[...]

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  3.8751080  0.7687276   5.041 2.50e-06 ***
mpg         -0.0109953  0.0085475  -1.286 0.201724
psngr       -0.1782818  0.0290618  -6.135 2.45e-08 ***
length       0.0067382  0.0032890   2.049 0.043502 *
width       -0.0517544  0.0151009  -3.427 0.000933 ***
weight       0.0008373  0.0001302   6.431 6.60e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
[...]
```

Briefly explain the information that is being provided in each column of the table. What are your conclusions and how would you try to improve the model?

Paper 1, Section II

13J Statistical Modelling

Consider a generalised linear model with parameter $\beta^\top$ partitioned as $(\beta_0^\top, \beta_1^\top)$, where β_0 has p_0 components and β_1 has $p - p_0$ components, and consider testing $H_0 : \beta_1 = 0$ against $H_1 : \beta_1 \neq 0$. Define carefully the deviance, and use it to construct a test for H_0.

[You may use Wilks' theorem to justify this test, and you may also assume that the dispersion parameter is known.]

Now consider the generalised linear model with Poisson responses and the canonical link function with linear predictor $\eta = (\eta_1, ..., \eta_n)^T$ given by $\eta_i = x_i^\top \beta$, $i = 1, ..., n$, where $x_{i1} = 1$ for every i. Derive the deviance for this model, and argue that it may be approximated by Pearson's χ^2 statistic.

Paper 4, Section II

13J Statistical Modelling

Every day, Barney the darts player comes to our laboratory. We record his facial expression, which can be either "mad", "weird" or "relaxed", as well as how many units of beer he has drunk that day. Each day he tries a hundred times to hit the bull's-eye, and we write down how often he succeeds. The data look like this:

```
>
Day Beer Expression  BullsEye
  1    3        Mad        30
  2    3        Mad        32
  ⋮    ⋮          ⋮         ⋮
 60    2        Mad        37
 61    4      Weird        30
  ⋮    ⋮          ⋮         ⋮
110    4      Weird        28
111    2    Relaxed        35
  ⋮    ⋮          ⋮         ⋮
150    3    Relaxed        31
```

Write down a reasonable model for $Y_1, \ldots, Y_n$, where $n = 150$ and where Y_i is the number of times Barney has hit bull's-eye on the ith day. Explain briefly why we may wish initially to include interactions between the variables. Write the R code to fit your model.

The scientist of the above story fitted her own generalized linear model, and subsequently obtained the following summary (abbreviated):

```
> summary(barney)
[...]

Coefficients:
                        Estimate Std. Error z value Pr(>|z|)
(Intercept)             -0.37258    0.05388  -6.916 4.66e-12 ***
Beer                    -0.09055    0.01595  -5.676 1.38e-08 ***
ExpressionWeird         -0.10005    0.08044  -1.244 0.213570
ExpressionRelaxed        0.29881    0.08268   3.614 0.000301 ***
Beer:ExpressionWeird     0.03666    0.02364   1.551 0.120933
Beer:ExpressionRelaxed -0.07697    0.02845  -2.705 0.006825 **

[...]
```

Why are `ExpressionMad` and `Beer:ExpressionMad` not listed? Suppose on a particular day, Barney's facial expression is weird, and he drank three units of beer. Give the linear predictor in the scientist's model for this day.

Based on the summary, how could you improve your model? How could one fit this new model in R (without modifying the data file)?

Paper 2, Section II

35C Statistical Physics

Consider a 3-dimensional gas of N non-interacting particles in a box of size L where the allowed momenta are $\{\mathbf{p}_i\}$. Assuming the particles have an energy $\epsilon(|\mathbf{p}|)$, $\epsilon'(p) > 0$, calculate the density of states $g(\epsilon)d\epsilon$ as $L \to \infty$.

Treating the particles as classical explain why the partition function is

$$Z = \frac{z^N}{N!}\,, \qquad z = \int_0^\infty d\epsilon\; g(\epsilon)\, e^{-\epsilon/kT}\,.$$

Obtain an expression for the total energy E.

Why is $\mathbf{p}_i \propto 1/L$? By considering the dependence of the energies on the volume V show that the pressure P is given by

$$PV = \frac{N}{3z}\int_0^\infty d\epsilon\; g(\epsilon)\, p\, \epsilon'(p)\, e^{-\epsilon/kT}\,.$$

What are the results for the pressure for non-relativistic particles and also for relativistic particles when their mass can be neglected?

What is the thermal wavelength for non-relativistic particles? Why are the classical results correct if the thermal wavelength is much smaller than the mean particle separation?

Paper 3, Section II
35C Statistical Physics

(i) Given the following density of states for a particle in 3 dimensions

$$g(\varepsilon) \;=\; KV\varepsilon^{1/2}$$

write down the partition function for a gas of N such non-interacting particles, assuming they can be treated classically. From this expression, calculate the energy E of the system and the heat capacities C_V and C_P. You may take it as given that $PV = \frac{2}{3}E$.

[*Hint: The formula* $\int_0^\infty dy\, y^2\, e^{-y^2} = \sqrt{\pi}/4$ *may be useful.*]

(ii) Using thermodynamic relations obtain the relation between heat capacities and compressibilities

$$\frac{C_P}{C_V} \;=\; \frac{\kappa_T}{\kappa_S}$$

where the isothermal and adiabatic compressibilities are given by

$$\kappa \;=\; -\frac{1}{V}\frac{\partial V}{\partial P}\,,$$

derivatives taken at constant temperature and entropy, respectively.

(iii) Find κ_T and κ_S for the ideal gas considered above.

Paper 4, Section II

34C Statistical Physics

(i) Let ρ_i be the probability that a system is in a state labelled by i with N_i particles and energy E_i. Define

$$s(\rho_i) = -k \sum_i \rho_i \log \rho_i \,.$$

$s(\rho_i)$ has a maximum, consistent with a fixed mean total number of particles N, mean total energy E and $\sum_i \rho_i = 1$, when $\rho_i = \bar{\rho}_i$. Let $S(E,N) = s(\bar{\rho}_i)$ and show that

$$\frac{\partial S}{\partial E} = \frac{1}{T}\,, \qquad \frac{\partial S}{\partial N} = -\frac{\mu}{T}\,,$$

where T may be identified with the temperature and μ with the chemical potential.

(ii) For two weakly coupled systems 1,2 then $\rho_{i,j} = \rho_{1,i}\,\rho_{2,j}$ and $E_{i,j} = E_{1,i} + E_{2,j}$, $N_{i,j} = N_{1,i} + N_{2,j}$. Show that $S(E,N) = S_1(E_1,N_1) + S_2(E_2,N_2)$ where, if $S(E,N)$ is stationary under variations in E_1, E_2 and N_1, N_2 for $E = E_1 + E_2$, $N = N_1 + N_2$ fixed, we must have $T_1 = T_2$, $\mu_1 = \mu_2$.

(iii) Define the grand partition function $\mathcal{Z}(T,\mu)$ for the system in (i) and show that

$$k \log \mathcal{Z} = S - \frac{1}{T}\,E + \frac{\mu}{T}\,N\,, \qquad S = \frac{\partial}{\partial T}\big(kT \log \mathcal{Z}\big)\,.$$

(iv) For a system with single particle energy levels ϵ_r the possible states are labelled by $i = \{n_r : n_r = 0,1\}$, where $N_i = \sum_r n_r$, $E_i = \sum_r n_r \epsilon_r$ and $\sum_i = \prod_r \sum_{n_r=0,1}$. Show that

$$\bar{\rho}_i = \prod_r \frac{e^{-n_r(\epsilon_r - \mu)/kT}}{1 + e^{-(\epsilon_r - \mu)/kT}}\,.$$

Calculate $\bar{n}_r$. How is this related to a free fermion gas?

Paper 1, Section II

29I Stochastic Financial Models

What is a Brownian motion? State the reflection principle for Brownian motion.

Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion. Let $M = \max_{0 \leqslant t \leqslant 1} W_t$. Prove

$$\mathbb{P}(M \geqslant x,\ W_1 \leqslant x - y) \;=\; \mathbb{P}(M \geqslant x,\ W_1 \geqslant x + y)$$

for all $x, y \geqslant 0$. Hence, show that the random variables M and $|W_1|$ have the same distribution.

Find the density function of the random variable $R = W_1/M$.

Paper 2, Section II

30I Stochastic Financial Models

What is a martingale? What is a supermartingale? What is a stopping time?

Let $M = (M_n)_{n \geqslant 0}$ be a martingale and $\hat{M} = (\hat{M}_n)_{n \geqslant 0}$ a supermartingale with respect to a common filtration. If $M_0 = \hat{M}_0$, show that $\mathbb{E}M_T \geqslant \mathbb{E}\hat{M}_T$ for any bounded stopping time T.

[If you use a general result about supermartingales, you must prove it.]

Consider a market with one stock with prices $S = (S_n)_{n \geqslant 0}$ and constant interest rate r. Explain why an investor's wealth X satisfies

$$X_n \;=\; (1+r)\,X_{n-1} + \pi_n[S_n - (1+r)\,S_{n-1}]$$

where π_n is the number of shares of the stock held during the nth period.

Given an initial wealth X_0, an investor seeks to maximize $\mathbb{E}U(X_N)$ where U is a given utility function. Suppose the stock price is such that $S_n = S_{n-1}\,\xi_n$ where $(\xi_n)_{n \geqslant 1}$ is a sequence of independent and identically distributed random variables. Let V be defined inductively by

$$V(n,x,s) \;=\; \sup_{p \in \mathbb{R}} \,\mathbb{E}\,V[n+1,\ (1+r)\,x - ps\,(1+r-\xi_1),\ s\,\xi_1]$$

with terminal condition $V(N,x,s) = U(x)$ for all $x, s \in \mathbb{R}$.

Show that the process $(V(n, X_n, S_n))_{0 \leqslant n \leqslant N}$ is a supermartingale for any trading strategy π.

Suppose π^* is a trading strategy such that the corresponding wealth process X^* makes $(V(n,\, X_n^*,\, S_n))_{0 \leqslant n \leqslant N}$ a martingale. Show that π^* is optimal.

Paper 3, Section II

29I Stochastic Financial Models

Consider a market with two assets, a riskless bond and a risky stock, both of whose initial (time-0) prices are $B_0 = 1 = S_0$. At time 1, the price of the bond is a constant $B_1 = R > 0$ and the price of the stock S_1 is uniformly distributed on the interval $[0, C]$ where $C > R$ is a constant.

Describe the set of state price densities.

Consider a contingent claim whose payout at time 1 is given by S_1^2. Use the fundamental theorem of asset pricing to show that, if there is no arbitrage, the initial price of the claim is larger than R and smaller than C.

Now consider an investor with initial wealth $X_0 = 1$, and assume $C = 3R$. The investor's goal is to maximize his expected utility of time-1 wealth $\mathbb{E}U[R + \pi(S_1 - R)]$, where $U(x) = \sqrt{x}$. Show that the optimal number of shares of stock to hold is $\pi^* = 1$.

What would be the investor's marginal utility price of the contingent claim described above?

Paper 4, Section II

29I Stochastic Financial Models

Consider a market with no riskless asset and d risky stocks where the price of stock $i \in \{1, \ldots, d\}$ at time $t \in \{0, 1\}$ is denoted S_t^i. We assume the vector $S_0 \in \mathbb{R}^d$ is not random, and we let $\mu = \mathbb{E}S_1$ and $V = \mathbb{E}\left[(S_1 - \mu)(S_1 - \mu)^T\right]$. Assume V is not singular.

Suppose an investor has initial wealth $X_0 = x$, which he invests in the d stocks so that his wealth at time 1 is $X_1 = \pi^T S_1$ for some $\pi \in \mathbb{R}^d$. He seeks to minimize the $\mathrm{var}(X_1)$ subject to his budget constraint and the condition that $\mathbb{E}X_1 = m$ for a given constant $m \in \mathbb{R}$.

Illustrate this investor's problem by drawing a diagram of the mean-variance efficient frontier. Write down the Lagrangian for the problem. Show that there are two vectors π_A and π_B (which do not depend on the constants x and m) such that the investor's optimal portfolio is a linear combination of π_A and π_B.

Another investor with initial wealth $Y_0 = y$ seeks to maximize $\mathbb{E}U(Y_1)$ his expected utility of time 1 wealth, subject to his budget constraint. Assuming that S_1 is Gaussian and $U(w) = -e^{-\gamma w}$ for a constant $\gamma > 0$, show that the optimal portfolio in this case is also a linear combination of π_A and π_B.

[You may use the moment generating function of the Gaussian distribution without derivation.]

Continue to assume S_1 is Gaussian, but now assume that U is increasing, concave, and twice differentiable, and that U, U' and U'' are of exponential growth but not necessarily of the form $U(w) = -e^{-\gamma w}$. (Recall that a function f is of exponential growth if $|f(w)| \leqslant a e^{b|w|}$ for some constants positive constants a, b.) Prove that the utility maximizing investor still holds a linear combination of π_A and π_B.

[You may use the Gaussian integration by parts formula

$$\mathbb{E}\left[\nabla f(Z)\right] = \mathbb{E}\left[Z f(Z)\right]$$

where $Z = (Z_1, \ldots, Z_d)^T$ is a vector of independent standard normal random variables, and f is differentiable and of exponential growth. You may also interchange integration and differentiation without justification.]

Paper 1, Section I

2F Topics in Analysis

Let (X, d) be a non-empty complete metric space with no isolated points, G an open dense subset of X and E a countable dense subset of X.

(i) Stating clearly any standard theorem you use, prove that $G \setminus E$ is a dense subset of X.

(ii) If G is only assumed to be uncountable and dense in X, does it still follow that $G \setminus E$ is dense in X? Justify your answer.

Paper 2, Section I

2F Topics in Analysis

(a) State the Weierstrass approximation theorem concerning continuous real functions on the closed interval $[0, 1]$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be continuous.

(i) If $\int_0^1 f(x)\, x^n\, dx = 0$ for each $n = 0, 1, 2, \ldots$, prove that f is the zero function.

(ii) If we only assume that $\int_0^1 f(x)\, x^{2n}\, dx = 0$ for each $n = 0, 1, 2, \ldots$, prove that it still follows that f is the zero function.

[If you use the Stone–Weierstrass theorem, you must prove it.]

(iii) If we only assume that $\int_0^1 f(x)\, x^{2n+1}\, dx = 0$ for each $n = 0, 1, 2, \ldots$, does it still follow that f is the zero function? Justify your answer.

Paper 3, Section I

2F Topics in Analysis

Let $A = \{z \in \mathbb{C} \,:\, 1/2 \leqslant |z| \leqslant 2\}$ and suppose that f is complex analytic on an open subset containing A.

(i) Give an example, with justification, to show that there need not exist a sequence of complex polynomials converging to f uniformly on A.

(ii) Let $R \subset \mathbb{C}$ be the positive real axis and $B = A \setminus R$. Prove that there exists a sequence of complex polynomials $p_1, p_2, p_3, \ldots$ such that $p_j \to f$ uniformly on each compact subset of B.

(iii) Let $p_1, p_2, p_3, \ldots$ be the sequence of polynomials in (ii). If this sequence converges uniformly on A, show that $\int_C f(z)\,dz = 0$, where $C = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

Paper 4, Section I

2F Topics in Analysis

Find explicitly a polynomial p of degree $\leqslant 3$ such that

$$\sup_{x\in[-1,1]} |\, x^4 - p(x)| \;\leqslant\; \sup_{x\in[-1,1]} |\, x^4 - q(x)|$$

for every polynomial q of degree $\leqslant 3$. Justify your answer.

Paper 2, Section II

11F Topics in Analysis

Let

$$B_r(0) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < r^2\}\,,$$

$B = B_1(0)$, and

$$C = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}\,.$$

Let $D = B \cup C$.

(i) State the Brouwer fixed point theorem on the plane.

(ii) Show that the Brouwer fixed point theorem on the plane is equivalent to the non-existence of a continuous map $F \,:\, D \to C$ such that $F(p) = p$ for each $p \in C$.

(iii) Let $G \,:\, D \to \mathbb{R}^2$ be continuous, $0 < \epsilon < 1$ and suppose that

$$|p - G(p)| \,<\, \epsilon$$

for each $p \in C$. Using the Brouwer fixed point theorem or otherwise, prove that

$$B_{1-\epsilon}(0) \,\subseteq\, G(B)\,.$$

[*Hint: argue by contradiction.*]

(iv) Let $q \in B$. Does there exist a continuous map $H \,:\, D \to \mathbb{R}^2 \setminus \{q\}$ such that $H(p) = p$ for each $p \in C$? Justify your answer.

Paper 3, Section II

12F Topics in Analysis

(i) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map with $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ of γ about the origin.

(ii) For $j = 0, 1$, let $\gamma_j : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous with $\gamma_j(0) = \gamma_j(1)$. Make the following statement precise, and prove it: *if γ_0 can be continuously deformed into γ_1 through a family of continuous curves missing the origin, then $w(\gamma_0; 0) = w(\gamma_1; 0)$.*

[You may use without proof the following fact: if $\gamma, \delta : [0,1] \to \mathbb{C} \setminus \{0\}$ are continuous with $\gamma(0) = \gamma(1)$, $\delta(0) = \delta(1)$ and if $|\gamma(t)| < |\delta(t)|$ for each $t \in [0,1]$, then $w(\gamma + \delta; 0) = w(\delta; 0)$.]

(iii) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous with $\gamma(0) = \gamma(1)$. If $\gamma(t)$ is not equal to a negative real number for each $t \in [0,1]$, prove that $w(\gamma; 0) = 0$.

(iv) Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. If $g : D \to C$ is continuous, prove that for each non-zero integer n, there is at least one point $z \in C$ such that $z^n + g(z) = 0$.

Paper 1, Section II

38A Waves

Derive the wave equation governing the velocity potential ϕ for linearized sound waves in a compressible inviscid fluid. How is the pressure disturbance related to the velocity potential?

A semi-infinite straight tube of uniform cross-section is aligned along the positive x-axis with its end at $x = -L$. The tube is filled with fluid of density ρ_1 and sound speed c_1 in $-L < x < 0$ and with fluid of density ρ_2 and sound speed c_2 in $x > 0$. A piston at the end of the tube performs small oscillations such that its position is $x = -L + \epsilon\, e^{i\omega t}$, with $\epsilon \ll L$ and $\epsilon\,\omega \ll c_1, c_2$. Show that the complex amplitude of the velocity potential in $x > 0$ is

$$-\epsilon\, c_1 \left(\frac{c_1}{c_2}\, \cos \frac{\omega L}{c_1} + i\, \frac{\rho_2}{\rho_1}\, \sin \frac{\omega L}{c_1} \right)^{-1} .$$

Calculate the time-averaged acoustic energy flux in $x > 0$. Comment briefly on the variation of this result with L for the particular case $\rho_2 \ll \rho_1$ and $c_2 = \mathrm{O}(c_1)$.

Paper 2, Section II

38A Waves

The equation of motion for small displacements $\mathbf{u}(\mathbf{x},t)$ in a homogeneous, isotropic, elastic medium of density ρ is

$$\rho\,\frac{\partial^2\mathbf{u}}{\partial t^2} \;=\; (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu\,\nabla^2\mathbf{u}\,,$$

where λ and μ are the Lamé constants. Show that the dilatation $\boldsymbol{\nabla}\cdot\mathbf{u}$ and rotation $\boldsymbol{\nabla}\wedge\mathbf{u}$ each satisfy wave equations, and determine the corresponding wave speeds c_P and c_S.

Show also that a solution of the form $\mathbf{u} = \mathbf{A}\exp\left[i(\mathbf{k}\cdot\mathbf{x}-\omega t)\right]$ satisfies

$$\omega^2\mathbf{A} \;=\; c_P^2\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{A}) - c_S^2\,\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{A})\,.$$

Deduce the dispersion relation and the direction of polarization relative to $\mathbf{k}$ for plane harmonic P-waves and plane harmonic S-waves.

Now suppose the medium occupies the half-space $z \leqslant 0$ and that the boundary $z = 0$ is stress free. Show that it is possible to find a self-sustained combination of evanescent P-waves and SV-waves (i.e. a Rayleigh wave), proportional to $\exp\left[ik(x-ct)\right]$ and propagating along the boundary, provided the wavespeed c satisfies

$$\left(2-\frac{c^2}{c_S^2}\right)^2 \;=\; 4\left(1-\frac{c^2}{c_S^2}\right)^{1/2}\left(1-\frac{c^2}{c_P^2}\right)^{1/2}\,.$$

[You are not required to show that this equation has a solution.]

Paper 3, Section II

38A Waves

Consider the equation

$$\frac{\partial^2 \phi}{\partial t \partial x} = -\alpha\, \phi\,,$$

where α is a positive constant. Find the dispersion relation for waves of frequency ω and wavenumber k. Sketch graphs of the phase velocity $c(k)$ and the group velocity $c_g(k)$.

A disturbance localized near $x = 0$ at $t = 0$ evolves into a dispersing wave packet. Will the wavelength and frequency of the waves passing a stationary observer located at a large positive value of x increase or decrease for $t > 0$? In which direction do the crests pass the observer?

Write down the solution $\phi(x, t)$ with initial value

$$\phi(x, 0) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk\,.$$

What can be said about $A(-k)$ if ϕ is real?

Use the method of stationary phase to obtain an approximation for $\phi(Vt, t)$ for fixed $V > 0$ and large t. What can be said about the solution at $x = -Vt$ for large t?

[You may assume that $\displaystyle\int_{-\infty}^{\infty} e^{-au^2}\, du = \sqrt{\frac{\pi}{a}}$ for $\mathrm{Re}(a) \geqslant 0$, $a \neq 0$.]

Paper 4, Section II

38A Waves

Starting from the equations for one-dimensional unsteady flow of an inviscid compressible fluid, show that it is possible to find Riemann invariants $u \pm Q$ that are constant on characteristics $C_{\pm}$ given by

$$\frac{dx}{dt} = u \pm c\,,$$

where $u(x,t)$ is the velocity of the fluid and $c(x,t)$ is the local speed of sound. Show that $Q = 2(c - c_0)/(\gamma - 1)$ for the case of a perfect gas with adiabatic equation of state $p = p_0(\rho/\rho_0)^{\gamma}$, where p_0 , ρ_0 and γ are constants, $\gamma > 1$ and $c = c_0$ when $\rho = \rho_0$.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas is initially uniform and at rest with density ρ_0 . At $t = 0$ the piston starts moving to the left at a constant speed V. Assuming that the gas keeps up with the piston, find $u(x,t)$ and $c(x,t)$ in each of the three distinct regions that are defined by families of C_+ characteristics.

Now assume that the gas does not keep up with the piston. Show that the gas particle at $x = x_0$ when $t = 0$ follows a trajectory given, for $t > x_0/c_0$, by

$$x(t) \;=\; \frac{\gamma+1}{\gamma-1}\left(\frac{c_0 t}{x_0}\right)^{2/(\gamma+1)} x_0 - \frac{2\,c_0 t}{\gamma-1}\,.$$

Deduce that the velocity of any given particle tends to $-2\,c_0/(\gamma - 1)$ as $t \to \infty$.