UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday, 2 June, 2010 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, ..., ***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Let p be an odd prime number. If n is an integer prime to p, define $\left(\frac{n}{p}\right)$.

(i) Prove that $\chi(n) = \left(\frac{n}{p}\right)$ defines a homomorphism from $(\mathbb{Z}/p\mathbb{Z})^\times$ to the group $\{\pm 1\}$. What is the value of $\chi(-1)$?

(ii) If $p \equiv 1 \bmod 4$, prove that $\displaystyle\sum_{n=1}^{p-1} \chi(n)\, n = 0$.

2F Topics in Analysis

(a) State the Weierstrass approximation theorem concerning continuous real functions on the closed interval $[0,1]$.

(b) Let $f : [0,1] \to \mathbb{R}$ be continuous.

(i) If $\int_0^1 f(x)\, x^n\, dx = 0$ for each $n = 0, 1, 2, \ldots$, prove that f is the zero function.

(ii) If we only assume that $\int_0^1 f(x)\, x^{2n}\, dx = 0$ for each $n = 0, 1, 2, \ldots$, prove that it still follows that f is the zero function.

[If you use the Stone–Weierstrass theorem, you must prove it.]

(iii) If we only assume that $\int_0^1 f(x)\, x^{2n+1}\, dx = 0$ for each $n = 0, 1, 2, \ldots$, does it still follow that f is the zero function? Justify your answer.

3F Geometry of Group Actions

Show that a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry for the Euclidean metric on the plane $\mathbb{R}^2$ if and only if there is a vector $\boldsymbol{v} \in \mathbb{R}^2$ and an orthogonal linear map $B \in \mathrm{O}(2)$ with

$$T(\boldsymbol{x}) = B(\boldsymbol{x}) + \boldsymbol{v} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^2 .$$

When T is an isometry with $\det B = -1$, show that T is either a reflection or a glide reflection.

4H Coding and Cryptography

Describe the standard Hamming code of length 7, proving that it corrects a single error. Find its weight enumeration polynomial.

5J Statistical Modelling

Suppose you have a parametric model consisting of probability mass functions $f(y;\theta)$, $\theta \in \Theta \subset \mathbb{R}$. Given a sample $Y_1, ..., Y_n$ from $f(y;\theta)$, define the maximum likelihood estimator $\hat{\theta}_n$ for θ and, assuming standard regularity conditions hold, state the asymptotic distribution of $\sqrt{n}\,(\hat{\theta}_n - \theta)$.

Compute the Fisher information of a single observation in the case where $f(y;\theta)$ is the probability mass function of a Poisson random variable with parameter θ. If $Y_1, ..., Y_n$ are independent and identically distributed random variables having a Poisson distribution with parameter θ, show that $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $S = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ are unbiased estimators for θ. Without calculating the variance of S, show that there is no reason to prefer S over $\overline{Y}$.

[You may use the fact that the asymptotic variance of $\sqrt{n}\,(\hat{\theta}_n - \theta)$ is a lower bound for the variance of any unbiased estimator.]

6A Mathematical Biology

The population of a certain species subjected to a specific kind of predation is modelled by the difference equation

$$u_{t+1} = a\,\frac{u_t^2}{b^2 + u_t^2}\,, \qquad a > 0\,.$$

Determine the equilibria and show that if $a^2 > 4\,b^2$ it is possible for the population to be driven to extinction if it becomes less than a critical size which you should find. Explain your reasoning by means of a cobweb diagram.

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = \mu\left(\frac{1}{3}\,x^3 - x\right) + y\,, \qquad \dot{y} = -x\,,$$

where the parameter $\mu > 0$. Using Lyapunov's approach, discuss the stability of the fixed point and its domain of attraction. Relevant definitions or theorems that you use should be stated carefully, but proofs are not required.

8E Further Complex Methods

Define

$$F^{\pm}(x) \;=\; \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-(x\pm i\epsilon)}\, dt\,, \qquad x\,\in\,\mathbb{R}\,.$$

Using the fact that

$$F^{\pm}(x) \;=\; \pm\,\frac{f(x)}{2} \;+\; \frac{1}{2\pi i}\, P\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt\,, \qquad x\,\in\,\mathbb{R}\,,$$

where P denotes the Cauchy principal value, find two complex-valued functions $F^{+}(z)$ and $F^{-}(z)$ which satisfy the following conditions

1. $F^{+}(z)$ and $F^{-}(z)$ are analytic for $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ respectively, $z = x + iy$;
2. $F^{+}(x) - F^{-}(x) \;=\; \dfrac{\sin x}{x}\,, \qquad x \in \mathbb{R}\,;$
3. $F^{\pm}(z) = \mathrm{O}\left(\frac{1}{z}\right)\,, \qquad z \to \infty\,, \qquad \operatorname{Im} z \neq 0\,.$

9D Classical Dynamics

Given the form

$$T \;=\; \frac{1}{2}\, T_{ij}\, \dot{q}_i\, \dot{q}_j\,, \qquad V \;=\; \frac{1}{2}\, V_{ij}\, q_i\, q_j\,,$$

for the kinetic energy T and potential energy V of a mechanical system, deduce Lagrange's equations of motion.

A light elastic string of length $4b$, fixed at both ends, has three particles, each of mass m, attached at distances $b, 2b, 3b$ from one end. Gravity can be neglected. The particles vibrate with small oscillations transversely to the string, the tension S in the string providing the restoring force. Take the displacements of the particles, q_i, $i = 1, 2, 3$, to be the generalized coordinates. Take units such that $m = 1$, $S/b = 1$ and show that

$$V \;=\; \frac{1}{2}\left[\, {q_1}^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + {q_3}^2 \,\right].$$

Find the normal–mode frequencies for this system.

10D Cosmology

The number density $n = N/V$ for a photon gas in equilibrium is given by

$$n = \frac{8\pi}{c^3} \int_0^\infty \frac{\nu^2}{e^{h\nu/kT} - 1}\, d\nu\,,$$

where ν is the photon frequency. By letting $x = h\nu/kT$, show that

$$n = \alpha T^3\,,$$

where α is a constant which need not be evaluated.

The photon entropy density is given by

$$s = \beta T^3\,,$$

where β is a constant. By considering the entropy, explain why a photon gas cools as the universe expands.

SECTION II

11F Topics in Analysis

Let

$$B_r(0) = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 < r^2\} \, ,$$

$B = B_1(0)$, and

$$C = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = 1\} \, .$$

Let $D = B \cup C$.

(i) State the Brouwer fixed point theorem on the plane.

(ii) Show that the Brouwer fixed point theorem on the plane is equivalent to the non-existence of a continuous map $F : D \to C$ such that $F(p) = p$ for each $p \in C$.

(iii) Let $G : D \to \mathbb{R}^2$ be continuous, $0 < \epsilon < 1$ and suppose that

$$|p - G(p)| < \epsilon$$

for each $p \in C$. Using the Brouwer fixed point theorem or otherwise, prove that

$$B_{1-\epsilon}(0) \subseteq G(B) \, .$$

[*Hint: argue by contradiction.*]

(iv) Let $q \in B$. Does there exist a continuous map $H : D \to \mathbb{R}^2 \setminus \{q\}$ such that $H(p) = p$ for each $p \in C$? Justify your answer.

12H Coding and Cryptography

The Van der Monde matrix $V(x_0, x_1, \ldots, x_{r-1})$ is the $r \times r$ matrix with (i,j)th entry x_{i-1}^{j-1}. Find an expression for $\det V(x_0, x_1, \ldots, x_{r-1})$ as a product. Explain why this expression holds if we work modulo p a prime.

Show that $\det V(x_0, x_1, \ldots, x_{r-1}) \equiv 0$ modulo p if $r > p$, and that there exist $x_0, \ldots, x_{p-1}$ such that $\det V(x_0, x_1, \ldots, x_{p-1}) \not\equiv 0$. By using Wilson's theorem, or otherwise, find the possible values of $\det V(x_0, x_1, \ldots, x_{p-1})$ modulo p.

The Dark Lord Y'Trinti has acquired the services of the dwarf Trigon who can engrave pairs of very large integers on very small rings. The Dark Lord wishes Trigon to engrave n rings in such a way that anyone who acquires r of the rings and knows the Prime Perilous p can deduce the Integer N of Power, but owning $r-1$ rings will give no information whatsoever. The integers N and p are very large and $p > N$. Advise the Dark Lord.

For reasons to be explained in the prequel, Trigon engraves an $(n+1)$st ring with random integers. A band of heroes (who know the Prime Perilous and all the information contained in this question) set out to recover the rings. What, if anything, can they say, with very high probability, about the Integer of Power if they have r rings (possibly including the fake)? What can they say if they have $r+1$ rings? What if they have $r+2$ rings?

13A Mathematical Biology

The radially symmetric spread of an insect population density $n(r,t)$ in the plane is described by the equation

$$\frac{\partial n}{\partial t} = \frac{D_0}{r}\frac{\partial}{\partial r}\left[r\left(\frac{n}{n_0}\right)^2\frac{\partial n}{\partial r}\right]. \qquad (*)$$

Suppose Q insects are released at $r=0$ at $t=0$. We wish to find a similarity solution to $(*)$ in the form

$$n(r,t) = \frac{n_0}{\lambda^2(t)}\,F\left(\frac{r}{r_0\lambda(t)}\right).$$

Show first that the PDE $(*)$ reduces to an ODE for F if $\lambda(t)$ obeys the equation

$$\lambda^5\,\frac{d\lambda}{dt} = C\,\frac{D_0}{r_0^2},$$

where C is an arbitrary constant (that may be set to unity), and then obtain $\lambda(t)$ and F such that $F(0)=1$ and $F(\xi)=0$ for $\xi \geqslant 1$. Determine r_0 in terms of n_0 and Q. Sketch the function $n(r,t)$ at various times to indicate its qualitative behaviour.

14E Further Complex Methods

Let

$$I(z) = i\oint_C \frac{u^{z-1}}{u^2-4u+1}\,du,$$

where C is a closed anti-clockwise contour which consists of the unit circle joined to a loop around a branch cut along the negative axis between -1 and 0. Show that

$$I(z) = F(z) + G(z),$$

where

$$F(z) = 2\sin(\pi z)\int_0^1 \frac{x^{z-1}}{x^2+4x+1}\,dx, \qquad \operatorname{Re} z > 0,$$

and

$$G(z) = \frac{1}{2}\int_{-\pi}^{\pi}\frac{e^{i(z-1)\theta}}{1+2\sin^2\frac{\theta}{2}}\,d\theta, \qquad z \in \mathbb{C}.$$

Evaluate $I(z)$ using Cauchy's theorem. Explain how this may be used to obtain an analytic continuation of $F(z)$ valid for all $z \in \mathbb{C}$.

15D Classical Dynamics

An axially–symmetric top of mass m is free to rotate about a fixed point O on its axis. The principal moments of inertia about O are A, A, C, and the centre of gravity G is at a distance ℓ from O. Define Euler angles θ, ϕ and ψ which specify the orientation of the top, where θ is the inclination of OG to the upward vertical. Show that there are three conserved quantities for the motion, and give their physical meaning.

Initially, the top is spinning with angular velocity n about OG, with G vertically above O, before being disturbed slightly. Show that, in the subsequent motion, θ will remain close to zero provided $C^2n^2 > 4mg\ell A$, but that if $C^2n^2 < 4mg\ell A$, then θ will attain a maximum value given by

$$\cos\theta \simeq (C^2n^2/2mg\ell A) - 1 \,.$$

16G Logic and Set Theory

Let α be a non-zero ordinal. Prove that there exists a greatest ordinal β such that $\omega^\beta \leqslant \alpha$. Explain why there exists an ordinal γ with $\omega^\beta + \gamma = \alpha$. Prove that γ is unique, and that $\gamma < \alpha$.

A non-zero ordinal α is called *decomposable* if it can be written as the sum of two smaller non-zero ordinals. Deduce that if α is not a power of ω then α is decomposable.

Conversely, prove that if α is a power of ω then α is not decomposable.

[*Hint: consider the cases* $\alpha = \omega^\beta$ *(β a successor) and* $\alpha = \omega^\beta$ *(β a limit) separately.*]

17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G contains a *matching* from X to Y?

State and prove Hall's Marriage Theorem, giving a necessary and sufficient condition for G to contain a matching from X to Y.

Now assume that G does contain a matching (from X to Y). For a subset $A \subset X$, $\Gamma(A)$ denotes the set of vertices adjacent to some vertex in A.

(i) Suppose $|\Gamma(A)| > |A|$ for every $A \subset X$ with $A \neq \emptyset$, X. Show that every edge of G is contained in a matching.

(ii) Suppose that every edge of G is contained in a matching and that G is connected. Show that $|\Gamma(A)| > |A|$ for every $A \subset X$ with $A \neq \emptyset$, X.

(iii) For each $n \geqslant 2$, give an example of G with $|X| = n$ such that every edge is contained in a matching but $|\Gamma(A)| = |A|$ for some $A \subset X$ with $A \neq \emptyset$, X.

(iv) Suppose that every edge of G is contained in a matching. Must every pair of independent edges in G be contained in a matching? Give a proof or counterexample as appropriate.

[No form of Menger's Theorem or of the Max-Flow-Min-Cut Theorem may be assumed without proof.]

18H Galois Theory

(1) Let $F = \mathbb{Q}(\sqrt[3]{5}\,, \sqrt{5}\,, i)$. What is the degree of $F/\mathbb{Q}$? Justify your answer.

(2) Let F be a splitting field of $X^4 - 5$ over $\mathbb{Q}$. Determine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$. Determine all the subextensions of $F/\mathbb{Q}$, expressing each in the form $\mathbb{Q}(x)$ or $\mathbb{Q}(x, y)$ for some $x, y \in F$.

[*Hint: If an automorphism ρ of a field X has order 2, then for every $x \in X$ the element $x + \rho(x)$ is fixed by ρ.*]

19F Representation Theory

Define the concepts of induction and restriction of characters. State and prove the Frobenius Reciprocity Theorem.

Let H be a subgroup of G and let $g \in G$. We write $\mathcal{C}(g)$ for the conjugacy class of g in G, and write $C_G(g)$ for the centraliser of g in G. Suppose that $H \cap \mathcal{C}(g)$ breaks up into m conjugacy classes of H, with representatives $x_1, x_2, \ldots, x_m$.

Let ψ be a character of H. Writing $\mathrm{Ind}_H^G(\psi)$ for the induced character, prove that

(i) if no element of $\mathcal{C}(g)$ lies in H, then $\mathrm{Ind}_H^G(\psi)(g) = 0$,

(ii) if some element of $\mathcal{C}(g)$ lies in H, then

$$\mathrm{Ind}_H^G(\psi)(g) \;=\; |C_G(g)| \sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|}\,.$$

Let $G = S_4$ and let $H = \langle a, b\rangle$, where $a = (1\;2\;3\;4)$ and $b = (1\;3)$. Identify H as a dihedral group and write down its character table. Restrict each G-conjugacy class to H and calculate the H-conjugacy classes contained in each restriction. Given a character ψ of H, express $\mathrm{Ind}_H^G(\psi)(g)$ in terms of ψ, where g runs through a set of conjugacy classes of G. Use your calculation to find the values of all the irreducible characters of H induced to G.

20G Number Fields

Calculate the class group of the field $\mathbb{Q}(\sqrt{-14}\,)$.

21H Algebraic Topology

Let G be the finitely presented group $G = \langle a, b \,|\, a^2\, b^3\, a^3\, b^2 = 1\rangle$. Construct a path connected space X with $\pi_1(X, x) \cong G$. Show that X has a unique connected double cover $\pi : Y \to X$, and give a presentation for $\pi_1(Y, y)$.

22H Linear Analysis

For a sequence $x = (x_1, x_2, \dots)$ with $x_j \in \mathbb{C}$ for all $j \geqslant 1$, let

$$\|x\|_\infty := \sup_{j \geqslant 1} |x_j|$$

and $\ell^\infty = \{x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \geqslant 1 \text{ and } \|x\|_\infty < \infty\}$.

a) Prove that ℓ^∞ is a Banach space.

b) Define

$$c_0 = \{x = (x_1, x_2, \dots) \in \ell^\infty : \lim_{j \to \infty} x_j = 0\}$$

and

$$\ell^1 = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \in \mathbb{N} \text{ and } \|x\|_1 = \sum_{\ell=1}^{\infty} |x_\ell| < \infty \right\}.$$

Show that c_0 is a closed subspace of ℓ^∞. Show that $c_0^* \simeq \ell^1$.

[*Hint: find an isometric isomorphism from ℓ^1 to c_0^*.*]

c) Let

$$c_{00} = \{x = (x_1, x_2, \dots) \in \ell^\infty : x_j = 0 \text{ for all } j \text{ large enough}\}.$$

Is c_{00} a closed subspace of ℓ^∞? If not, what is the closure of c_{00}?

23G Riemann Surfaces

Given a complete analytic function $\mathcal{F}$ on a domain $U \subset \mathbb{C}$, describe *briefly* how the space of germs construction yields a Riemann surface R associated to $\mathcal{F}$ together with a covering map $\pi : R \to U$ (*proofs not required*).

In the case when π is regular, explain briefly how, given a point $P \in U$, any closed curve in U with initial and final points P yields a permutation of the set $\pi^{-1}(P)$.

Now consider the Riemann surface R associated with the complete analytic function

$$(z^2 - 1)^{1/2} + (z^2 - 4)^{1/2}$$

on $U = \mathbb{C} \setminus \{\pm 1, \pm 2\}$, with regular covering map $\pi : R \to U$. Which subgroup of the full symmetric group of $\pi^{-1}(P)$ is obtained in this way from all such closed curves (with initial and final points P)?

24G Algebraic Geometry

Let $X = X_{n,m,r}$ be the set of $n \times m$ matrices of rank at most r over a field k. Show that $X_{n,m,r}$ is naturally an affine subvariety of $\mathbf{A}^{nm}$ and that $X_{n,m,r}$ is a Zariski closed subvariety of $X_{n,m,r+1}$.

Show that if $r < \min(n,m)$, then 0 is a singular point of X.

Determine the dimension of $X_{5,2,1}$.

25H Differential Geometry

(i) State and prove the isoperimetric inequality for plane curves. You may appeal to Wirtinger's inequality as long as you state it precisely.

(ii) State Fenchel's theorem for curves in space.

(iii) Let $\alpha : I \to \mathbb{R}^2$ be a closed regular plane curve bounding a region K. Suppose $K \supset [p_1, p_1 + d_1] \times [p_2, p_2 + d_2]$, for $d_1 > 0$, $d_2 > 0$, i.e. K contains a rectangle of dimensions d_1, d_2. Let $k(s)$ denote the signed curvature of α with respect to the inward pointing normal, where α is parametrised anticlockwise. Show that there exists an $s_0 \in I$ such that $k(s_0) \leqslant \sqrt{\pi/(d_1 d_2)}$.

26I Probability and Measure

Show that any two probability measures which agree on a π-system also agree on the σ-algebra generated by that π-system.

State Fubini's theorem for non-negative measurable functions.

Let μ denote Lebesgue measure on $\mathbb{R}^2$. Fix $s \in [0,1)$. Set $c = \sqrt{1-s^2}$ and $\lambda = \sqrt{c}$. Consider the linear maps $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (\lambda^{-1}x,\, \lambda y), \qquad g(x,y) = (x,\, sx + y), \qquad h(x,y) = (x - sy,\, y).$$

Show that $\mu = \mu \circ f^{-1}$ and that $\mu = \mu \circ g^{-1}$. You must justify any assertion you make concerning the values taken by μ.

Compute $r = f \circ h \circ g \circ f$. Deduce that μ is invariant under rotations.

27I Applied Probability

(a) Let S_k be the sum of k independent exponential random variables of rate $k\mu$. Compute the moment generating function $\phi_{S_k}(\theta) = \mathbb{E}e^{\theta S_k}$ of S_k. Show that, as $k \to \infty$, functions $\phi_{S_k}(\theta)$ converge to a limit. Describe the random variable S for which the limiting function $\lim_{k\to\infty} \phi_{S_k}(\theta)$ coincides with $\mathbb{E}e^{\theta S}$.

(b) Define the M/G/1 queue with infinite capacity (sometimes written M/G/1/∞). Introduce the embedded discrete-time Markov chain (X_n) and write down the recursive relation between X_n and X_{n-1}.

Consider, for each fixed k and for $0 < \lambda < \mu$, an M/G/1/∞ queue with arrival rate λ and with service times distributed as S_k. Assume that the queue is empty at time 0. Let T_k be the earliest time at which a customer departs leaving the queue empty. Let A be the first arrival time and $B_k = T_k - A$ the length of the busy period.

(c) Prove that the moment generating functions $\phi_{B_k}(\theta) = \mathbb{E}e^{\theta B_k}$ and $\phi_{S_k}(\theta)$ are related by the equation

$$\phi_{B_k}(\theta) = \phi_{S_k}\left(\theta - \lambda\big(1 - \phi_{B_k}(\theta)\big)\right),$$

(d) Prove that the moment generating functions $\phi_{T_k}(\theta) = \mathbb{E}e^{\theta T_k}$ and $\phi_{S_k}(\theta)$ are related by the equation

$$\frac{\lambda - \theta}{\lambda}\phi_{T_k}(\theta) = \phi_{S_k}\left((\lambda - \theta)\big(\phi_{T_k}(\theta) - 1\big)\right).$$

(e) Assume that, for all $\theta < \lambda$,

$$\lim_{k\to\infty} \phi_{B_k}(\theta) = \mathbb{E}e^{\theta B}, \quad \lim_{k\to\infty} \phi_{T_k}(\theta) = \mathbb{E}e^{\theta T},$$

for some random variables B and T. Calculate $\mathbb{E}B$ and $\mathbb{E}T$. What service time distribution do these values correspond to?

28J Principles of Statistics

Define the *Kolmogorov–Smirnov* statistic for testing the null hypothesis that real random variables $X_1, \ldots, X_n$ are independently and identically distributed with specified continuous, strictly increasing distribution function F, and show that its null distribution does not depend on F.

A composite hypothesis H_0 specifies that, when the unknown positive parameter Θ takes value θ, the random variables $X_1, \ldots, X_n$ arise independently from the uniform distribution U $[0, \theta]$. Letting $J := \arg\max_{1 \leqslant i \leqslant n} X_i$, show that, under H_0, the statistic (J, X_J) is sufficient for Θ. Show further that, given $\{J = j,\ X_j = \xi\}$, the random variables $(X_i : i \neq j)$ are independent and have the U $[0, \xi]$ distribution. How might you apply the Kolmogorov–Smirnov test to test the hypothesis H_0?

29J Optimization and Control

(a) Suppose that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} V_{XX} & V_{XY} \\ V_{YX} & V_{YY} \end{pmatrix} \right).$$

Prove that conditional on $Y = y$, the distribution of X is again multivariate normal, with mean $\mu_X + V_{XY} V_{YY}^{-1}(y - \mu_Y)$ and covariance $V_{XX} - V_{XY} V_{YY}^{-1} V_{YX}$.

(b) The $\mathbb{R}^d$-valued process X evolves in discrete time according to the dynamics

$$X_{t+1} = AX_t + \varepsilon_{t+1},$$

where A is a constant $d \times d$ matrix, and ε_t are independent, with common $N(0, \Sigma_\varepsilon)$ distribution. The process X is not observed directly; instead, all that is seen is the process Y defined as

$$Y_t = CX_t + \eta_t,$$

where η_t are independent of each other and of the ε_t, with common $N(0, \Sigma_\eta)$ distribution.

If the observer has the prior distribution $X_0 \sim N(\hat{X}_0, V_0)$ for X_0, prove that at all later times the distribution of X_t conditional on $\mathcal{Y}_t \equiv (Y_1, \ldots, Y_t)$ is again normally distributed, with mean $\hat{X}_t$ and covariance V_t which evolve as

$$\begin{aligned} \hat{X}_{t+1} &= A\hat{X}_t + M_t C^T (\Sigma_\eta + C M_t C^T)^{-1} (Y_{t+1} - CA\hat{X}_t), \\ V_{t+1} &= M_t - M_t C^T (\Sigma_\eta + C M_t C^T)^{-1} C M_t, \end{aligned}$$

where

$$M_t = AV_t A^T + \Sigma_\varepsilon.$$

(c) In the special case where both X and Y are one-dimensional, and $A = C = 1$, $\Sigma_\varepsilon = 0$, find the form of the updating recursion. Show in particular that

$$\frac{1}{V_{t+1}} = \frac{1}{V_t} + \frac{1}{\Sigma_\eta}$$

and that

$$\frac{\hat{X}_{t+1}}{V_{t+1}} = \frac{\hat{X}_t}{V_t} + \frac{Y_{t+1}}{\Sigma_\eta}.$$

Hence deduce that, with probability one,

$$\lim_{t \to \infty} \hat{X}_t = \lim_{t \to \infty} t^{-1} \sum_{j=1}^{t} Y_j.$$

30I Stochastic Financial Models

What is a martingale? What is a supermartingale? What is a stopping time?

Let $M = (M_n)_{n \geqslant 0}$ be a martingale and $\hat{M} = (\hat{M}_n)_{n \geqslant 0}$ a supermartingale with respect to a common filtration. If $M_0 = \hat{M}_0$, show that $\mathbb{E}M_T \geqslant \mathbb{E}\hat{M}_T$ for any bounded stopping time T.

[If you use a general result about supermartingales, you must prove it.]

Consider a market with one stock with prices $S = (S_n)_{n \geqslant 0}$ and constant interest rate r. Explain why an investor's wealth X satisfies

$$X_n = (1+r)\,X_{n-1} + \pi_n[S_n - (1+r)\,S_{n-1}]$$

where π_n is the number of shares of the stock held during the nth period.

Given an initial wealth X_0, an investor seeks to maximize $\mathbb{E}U(X_N)$ where U is a given utility function. Suppose the stock price is such that $S_n = S_{n-1}\,\xi_n$ where $(\xi_n)_{n \geqslant 1}$ is a sequence of independent and identically distributed random variables. Let V be defined inductively by

$$V(n,x,s) = \sup_{p \in \mathbb{R}} \mathbb{E}\,V[n+1,\,(1+r)\,x - ps\,(1+r-\xi_1),\,s\,\xi_1]$$

with terminal condition $V(N,x,s) = U(x)$ for all $x, s \in \mathbb{R}$.

Show that the process $(V(n,X_n,S_n))_{0 \leqslant n \leqslant N}$ is a supermartingale for any trading strategy π.

Suppose π^* is a trading strategy such that the corresponding wealth process X^* makes $(V(n,\,X^*_n,\,S_n))_{0 \leqslant n \leqslant N}$ a martingale. Show that π^* is optimal.

31E Partial Differential Equations

(a) State the Lax-Milgram lemma. Use it to prove that there exists a unique function u in the space

$$H^2_\partial(\Omega) \;=\; \left\{u \in H^2(\Omega)\,;\, u|_{\partial\Omega} \;=\; \partial u/\partial\gamma|_{\partial\Omega} \;=\; 0\right\}\;,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and γ its outwards unit normal vector, which is the weak solution of the equations

$$\begin{aligned}\Delta^2 u &= f \quad \text{in} \quad \Omega\,,\\ u &= \frac{\partial u}{\partial\gamma} \;=\; 0 \quad \text{on} \quad \partial\,\Omega\,,\end{aligned}$$

for $f \in L^2(\Omega)$, Δ the Laplacian and $\Delta^2 = \Delta\Delta$.

[*Hint: Use regularity of the solution of the Dirichlet problem for the Poisson equation.*]

(b) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $u \in H^1(\Omega)$ and denote

$$\bar{u} \;=\; \int_\Omega u\, d^n x \Big/ \int_\Omega d^n x\;.$$

The following Poincaré-type inequality is known to hold

$$\|u - \bar{u}\|_{L^2} \;\leqslant\; C\|\nabla u\|_{L^2}\;,$$

where C only depends on Ω. Use the Lax-Milgram lemma and this Poincaré-type inequality to prove that the Neumann problem

$$\begin{aligned}\Delta u &= f \quad \text{in} \quad \Omega\,,\\ \frac{\partial u}{\partial\gamma} &= 0 \quad \text{on} \quad \partial\,\Omega\,,\end{aligned}$$

has a unique weak solution in the space

$$H^1_-(\Omega) \;=\; H^1(\Omega) \cap \{u : \Omega \to \mathbb{R}\,;\, \bar{u} \;=\; 0\}$$

if and only if $\bar{f} = 0$.

32E Integrable Systems

Consider the Gelfand–Levitan–Marchenko (GLM) integral equation

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,z)F(z+y)\,dz \;=\; 0\,,$$

with $F(x) = \sum_1^N \beta_n\, e^{-c_n\, x}$, where $c_1, \ldots, c_N$ are positive constants and $\beta_1, \ldots, \beta_N$ are constants. Consider separable solutions of the form

$$K(x,y) \;=\; \sum_{n=1}^{N} K_n(x)\, e^{-c_n\, y}\,,$$

and reduce the GLM equation to a linear system

$$\sum_{m=1}^{N} A_{nm}(x) K_m(x) \;=\; B_n(x)\,,$$

where the matrix $A_{nm}(x)$ and the vector $B_n(x)$ should be determined.

How is K related to solutions of the KdV equation?

Set $N = 1$, $c_1 = c$, $\beta_1 = \beta \exp(8c^3 t)$ where c, β are constants. Show that the corresponding one–soliton solution of the KdV equation is given by

$$u(x,t) \;=\; -\,\frac{4\,\beta_1 c\, e^{-2\,cx}}{(1 + (\beta_1/2c)\, e^{-2\,cx})^2}\,.$$

[You may use any facts about the Inverse Scattering Transform without proof.]

33C Principles of Quantum Mechanics

Consider a joint eigenstate of $\mathbf{J}^2$ and J_3, $|j\ m\rangle$. Write down a unitary operator $U(\mathbf{n},\theta)$ for rotation of the state by an angle θ about an axis with direction $\mathbf{n}$, where $\mathbf{n}$ is a unit vector. How would a state with zero orbital angular momentum transform under such a rotation?

What is the relation between the angular momentum operator $\mathbf{J}$ and the Pauli matrices $\boldsymbol{\sigma}$ when $j = \frac{1}{2}$? Explicitly calculate $(\mathbf{J}\cdot\mathbf{a})^2$, for an arbitrary real vector $\mathbf{a}$, in this case. What are the eigenvalues of the operator $\mathbf{J}\cdot\mathbf{a}$? Show that the unitary rotation operator for $j = \frac{1}{2}$ can be expressed as

$$U(\mathbf{n},\theta) = \cos\frac{\theta}{2} - i\,\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}\,. \qquad (*)$$

Starting with a state $|\frac{1}{2}\ m\rangle$ the component of angular momentum along a direction $\mathbf{n}'$, making and angle θ with the z-axis, is susequently measured to be m'. Immediately after this measurement the state is $|\frac{1}{2}\ m'\rangle_\theta$. Write down an eigenvalue equation for $|\frac{1}{2}\ m'\rangle_\theta$ in terms of $\mathbf{n}'\cdot\mathbf{J}$. Show that the probability for measuring an angular momentum of $m'\hbar$ along the direction $\mathbf{n}'$ is, assuming $\mathbf{n}'$ is in the x-z plane,

$$\left|\langle\tfrac{1}{2}\ m|\tfrac{1}{2}\ m'\rangle_\theta\right|^2 = \left|\langle\tfrac{1}{2}\ m|U(\mathbf{y},\theta)|\tfrac{1}{2}\ m'\rangle\right|^2,$$

where $\mathbf{y}$ is a unit vector in the y-direction. Using $(*)$ show that the probability that $m = +\frac{1}{2}$, $m' = -\frac{1}{2}$ is of the form

$$A + B\cos^2\frac{\theta}{2}\,,$$

determining the integers A and B in the process.

[Assume $\hbar = 1$. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
]

34B Applications of Quantum Mechanics

A beam of particles of mass m and momentum $p = \hbar k$ is incident along the z-axis. Write down the asymptotic form of the wave function which describes scattering under the influence of a spherically symmetric potential $V(r)$ and which defines the scattering amplitude $f(\theta)$.

Given that, for large r,

$$e^{ikr\cos\theta} \sim \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1) \left(e^{ikr} - (-1)^l \, e^{-ikr} \right) P_l(\cos\theta) \,,$$

show how to derive the partial-wave expansion of the scattering amplitude in the form

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \, e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta) \,.$$

Obtain an expression for the total cross-section, σ.

Let $V(r)$ have the form

$$V(r) = \begin{cases} -V_0 \,, & r < a \,, \\ 0 \,, & r > a \,, \end{cases}$$

where $V_0 = \dfrac{\hbar^2}{2m} \gamma^2$.

Show that the $l = 0$ phase-shift δ_0 satisfies

$$\frac{\tan(ka + \delta_0)}{ka} = \frac{\tan \kappa a}{\kappa a} \,,$$

where $\kappa^2 = k^2 + \gamma^2$.

Assume γ to be large compared with k so that κ may be approximated by γ. Show, using graphical methods or otherwise, that there are values for k for which $\delta_0(k) = n\pi$ for some integer n, which should not be calculated. Show that the smallest value, k_0, of k for which this condition holds certainly satisfies $k_0 < 3\pi/2a$.

35C Statistical Physics

Consider a 3-dimensional gas of N non-interacting particles in a box of size L where the allowed momenta are $\{\mathbf{p}_i\}$. Assuming the particles have an energy $\epsilon(|\mathbf{p}|)$, $\epsilon'(p) > 0$, calculate the density of states $g(\epsilon)d\epsilon$ as $L \to \infty$.

Treating the particles as classical explain why the partition function is

$$Z = \frac{z^N}{N!}\,, \qquad z = \int_0^\infty d\epsilon\; g(\epsilon)\, e^{-\epsilon/kT}\,.$$

Obtain an expression for the total energy E.

Why is $\mathbf{p}_i \propto 1/L$? By considering the dependence of the energies on the volume V show that the pressure P is given by

$$PV = \frac{N}{3z}\int_0^\infty d\epsilon\; g(\epsilon)\, p\, \epsilon'(p)\, e^{-\epsilon/kT}\,.$$

What are the results for the pressure for non-relativistic particles and also for relativistic particles when their mass can be neglected?

What is the thermal wavelength for non-relativistic particles? Why are the classical results correct if the thermal wavelength is much smaller than the mean particle separation?

36B General Relativity

A vector field k^a which satisfies

$$k_{a;b} + k_{b;a} = 0$$

is called a Killing vector field. Prove that k^a is a Killing vector field if and only if

$$k^c g_{ab,c} + k^c{}_{,b}\, g_{ac} + k^c{}_{,a}\, g_{bc} = 0\,.$$

Prove also that if V^a satisfies

$$V^a{}_{;b} V^b = 0\,,$$

then

$$(V^a k_a)_{,b} V^b = 0 \tag{$*$}$$

for any Killing vector field k^a.

In the two–dimensional space–time with coordinates $x^a = (u, v)$ and line element

$$ds^2 = -du^2 + u^2 dv^2\,,$$

verify that $(0, 1)$, $e^{-v}(1, u^{-1})$ and $e^v(-1, u^{-1})$ are Killing vector fields. Show, by using $(*)$ with V^a the tangent vector to a geodesic, that geodesics in this space–time are given by

$$\alpha e^v + \beta e^{-v} = 2\,\gamma u^{-1}\,,$$

where α, β and γ are arbitrary real constants.

37A Fluid Dynamics II

What is lubrication theory? Explain the assumptions that go into the theory.

Viscous fluid with dynamic viscosity μ and density ρ is contained between two flat plates, which approach each other at uniform speed V. The first is fixed at $y = 0$, $-L < x < L$. The second has its ends at $(-L,\, h_0 - \Delta h - Vt)$, $(L,\, h_0 + \Delta h - Vt)$, where $\Delta h \sim h_0 \ll L$. There is no flow in the z direction, and all variation in z may be neglected. There is no applied pressure gradient in the x direction.

Assuming that V is so small that lubrication theory applies, derive an expression for the horizontal volume flux $Q(x)$ at $t = 0$, in terms of the pressure gradient. Show that mass conservation implies that $dQ/dx = V$, so that $Q(L) - Q(-L) = 2VL$. Derive another relation between $Q(L)$ and $Q(-L)$ by setting the pressures at $x = \pm L$ to be equal, and hence show that

$$Q\,(\pm L) \;=\; VL\left(\frac{\Delta h}{h_0} \pm 1\right) .$$

Show that lubrication theory applies if $V \ll \mu/h_0\rho$.

38A Waves

The equation of motion for small displacements $\mathbf{u}(\mathbf{x}, t)$ in a homogeneous, isotropic, elastic medium of density ρ is

$$\rho\,\frac{\partial^2\mathbf{u}}{\partial t^2} \;=\; (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu\,\nabla^2\mathbf{u}\,,$$

where λ and μ are the Lamé constants. Show that the dilatation $\boldsymbol{\nabla}\cdot\mathbf{u}$ and rotation $\boldsymbol{\nabla}\wedge\mathbf{u}$ each satisfy wave equations, and determine the corresponding wave speeds c_P and c_S.

Show also that a solution of the form $\mathbf{u} = \mathbf{A}\exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$ satisfies

$$\omega^2\mathbf{A} \;=\; c_P^2\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{A}) - c_S^2\,\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{A})\,.$$

Deduce the dispersion relation and the direction of polarization relative to $\mathbf{k}$ for plane harmonic P-waves and plane harmonic S-waves.

Now suppose the medium occupies the half-space $z \leqslant 0$ and that the boundary $z = 0$ is stress free. Show that it is possible to find a self-sustained combination of evanescent P-waves and SV-waves (i.e. a Rayleigh wave), proportional to $\exp\left[ik(x - ct)\right]$ and propagating along the boundary, provided the wavespeed c satisfies

$$\left(2 - \frac{c^2}{c_S^2}\right)^2 = 4\left(1 - \frac{c^2}{c_S^2}\right)^{1/2}\left(1 - \frac{c^2}{c_P^2}\right)^{1/2} .$$

[You are not required to show that this equation has a solution.]

39A Numerical Analysis

The inverse discrete Fourier transform $\mathcal{F}_n^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formula

$$\mathbf{x} = \mathcal{F}_n^{-1}\,\mathbf{y}\,, \qquad \text{where} \qquad x_\ell = \sum_{j=0}^{n-1} \omega_n^{j\ell}\, y_j\,, \qquad \ell = 0, \ldots, n-1\,.$$

Here, $\omega_n = \exp(2\pi i/n)$ is the primitive root of unity of degree n, and $n = 2^p$, $p = 1, 2, \ldots$.

(1) Show how to assemble $\mathbf{x} = \mathcal{F}_{2m}^{-1}\,\mathbf{y}$ in a small number of operations if we already know the Fourier transforms of the even and odd portions of $\mathbf{y}$:

$$\mathbf{x}^{(\mathrm{E})} = \mathcal{F}_m^{-1}\,\mathbf{y}^{(\mathrm{E})}\,, \qquad \mathbf{x}^{(\mathrm{O})} = \mathcal{F}_m^{-1}\,\mathbf{y}^{(\mathrm{O})}\,.$$

(2) Describe the Fast Fourier Transform (FFT) method for evaluating $\mathbf{x}$ and draw a relevant diagram for $n = 8$.

(3) Find the costs of the FFT for $n = 2^p$ (only multiplications count).

(4) For $n = 4$, using the FFT technique, find

$$\mathbf{x} = \mathcal{F}_4^{-1}\,\mathbf{y}\,, \qquad \text{for} \qquad \mathbf{y} = [1, 1, -1, -1]\,, \qquad \text{and} \qquad \mathbf{y} = [1, -1, 1, -1]\,.$$

END OF PAPER