UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 31 May, 2010 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, ...,* ***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

(i) Let N be an integer $\geqslant 2$. Define the addition and multiplication on the set of congruence classes modulo N.

(ii) Let an integer $M \geqslant 1$ have expansion to the base 10 given by $a_s \ldots a_0$. Prove that 11 divides M if and only if $\sum_{i=0}^{s} (-1)^i a_i$ is divisible by 11.

2F Topics in Analysis

Let (X, d) be a non-empty complete metric space with no isolated points, G an open dense subset of X and E a countable dense subset of X.

(i) Stating clearly any standard theorem you use, prove that $G \setminus E$ is a dense subset of X.

(ii) If G is only assumed to be uncountable and dense in X, does it still follow that $G \setminus E$ is dense in X? Justify your answer.

3F Geometry of Group Actions

Explain what it means to say that G is a *crystallographic group* of isometries of the Euclidean plane and that $\overline{G}$ is its *point group*. Prove the *crystallographic restriction:* a rotation in such a point group $\overline{G}$ must have order $1, 2, 3, 4$ or 6.

4H Coding and Cryptography

Explain what is meant by saying that a binary code $\mathcal{C}$ is a decodable code with words C_j of length l_j for $1 \leqslant j \leqslant n$. Prove the MacMillan inequality which states that, for such a code,

$$\sum_{j=1}^{n} 2^{-l_j} \leqslant 1 \, .$$

5J Statistical Modelling

Consider a binomial generalised linear model for data $y_1, ..., y_n$ modelled as realisations of independent $Y_i \sim \text{Bin}(1, \mu_i)$ and logit link $\mu_i = e^{\beta x_i}/(1 + e^{\beta x_i})$ for some known constants x_i, $i = 1, \ldots, n$, and unknown scalar parameter β. Find the log-likelihood for β, and the likelihood equation that must be solved to find the maximum likelihood estimator $\hat{\beta}$ of β. Compute the second derivative of the log-likelihood for β, and explain the algorithm you would use to find $\hat{\beta}$.

6A Mathematical Biology

A delay model for a population N_t consists of

$$N_{t+1} = \frac{rN_t}{1 + b\,N_{t-1}^2}\,,$$

where t is discrete time, $r > 1$ and $b > 0$. Investigate the linear stability about the positive steady state N^*. Show that $r = 2$ is a bifurcation value at which the steady state bifurcates to a periodic solution of period 6.

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = -\mu x + y\,, \qquad \dot{y} = \frac{x^2}{1 + x^2} - \nu y\,,$$

where $x(t)$ and $y(t)$ are non-negative, the parameters μ and ν are strictly positive and $\mu \neq \nu$. Sketch the nullclines in the x, y plane. Deduce that for $\mu < \mu_c$ (where μ_c is to be determined) there are three fixed points. Find them and determine their type.

Sketch the phase portrait for $\mu < \mu_c$ and identify, qualitatively on your sketch, the stable and unstable manifolds of the saddle point. What is the final outcome of this system?

8E Further Complex Methods

Let the complex-valued function $f(z)$ be analytic in the neighbourhood of the point z_0 and let $u(x,y)$ be the real part of $f(z)$. Show that

$$f(z) = 2\,u\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) - \overline{f(z_0)}, \qquad z = x+iy.$$

Hence find the analytic function whose real part is

$$e^{-y}[\,x\cos x - y\sin x\,].$$

9D Classical Dynamics

A system with coordinates q_i, $i = 1,\ldots,n$, has the Lagrangian $L(q_i,\dot{q}_i)$. Define the energy E.

Consider a charged particle, of mass m and charge e, moving with velocity $\mathbf{v}$ in the presence of a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. The usual vector equation of motion can be derived from the Lagrangian

$$L = \frac{1}{2}m\,\mathbf{v}^2 + e\,\mathbf{v}\cdot\mathbf{A},$$

where $\mathbf{A}$ is the vector potential.

The particle moves in the presence of a field such that

$$\mathbf{A} = (0,\, r\,g(z),\, 0), \qquad g(z) > 0,$$

referred to cylindrical polar coordinates (r,ϕ,z). Obtain two constants of the motion, and write down the Lagrangian equations of motion obtained by variation of r,ϕ and z.

Show that, if the particle is projected from the point (r_0,ϕ_0,z_0) with velocity $(0,-2\,(e/m)\,r_0\,g(z_0),0)$, it will describe a circular orbit provided that $g'(z_0)=0$.

10D Cosmology

What is meant by the expression 'Hubble time'?

For $a(t)$ the scale factor of the universe and assuming $a(0) = 0$ and $a(t_0) = 1$, where t_0 is the time now, obtain a formula for the size of the particle horizon R_0 of the universe.

Taking

$$a(t) = (t/t_0)^{\alpha},$$

show that R_0 is finite for certain values of α. What might be the physically relevant values of α? Show that the age of the universe is less than the Hubble time for these values of α.

SECTION II

11F Geometry of Group Actions

For which circles Γ does inversion in Γ interchange 0 and ∞?

Let Γ be a circle that lies entirely within the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let K be inversion in this circle Γ, let J be inversion in the unit circle, and let T be the Möbius transformation $K \circ J$. Show that, if z_0 is a fixed point of T, then

$$J(z_0) = K(z_0)$$

and this point is another fixed point of T.

By applying a suitable isometry of the hyperbolic plane $\mathbb{D}$, or otherwise, show that Γ is the set of points at a fixed hyperbolic distance from some point of $\mathbb{D}$.

12H Coding and Cryptography

State and prove Shannon's theorem for the capacity of a noisy memoryless binary symmetric channel, defining the terms you use.

[You may make use of any form of Stirling's formula and any standard theorems from probability, provided that you state them exactly.]

13J Statistical Modelling

Consider a generalised linear model with parameter $\beta^\top$ partitioned as $(\beta_0^\top, \beta_1^\top)$, where β_0 has p_0 components and β_1 has $p - p_0$ components, and consider testing $H_0 : \beta_1 = 0$ against $H_1 : \beta_1 \neq 0$. Define carefully the deviance, and use it to construct a test for H_0.

[You may use Wilks' theorem to justify this test, and you may also assume that the dispersion parameter is known.]

Now consider the generalised linear model with Poisson responses and the canonical link function with linear predictor $\eta = (\eta_1, ..., \eta_n)^T$ given by $\eta_i = x_i^\top \beta$, $i = 1, ..., n$, where $x_{i1} = 1$ for every i. Derive the deviance for this model, and argue that it may be approximated by Pearson's χ^2 statistic.

14E Further Complex Methods

Consider the partial differential equation for $u(x,t)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \beta\,\frac{\partial u}{\partial x}\,, \quad \beta > 0\,, \quad 0 < x < \infty\,, \quad t > 0\,, \qquad (*)$$

where $u(x,t)$ is required to vanish rapidly for all t as $x \to \infty$.

(i) Verify that the PDE $(*)$ can be written in the following form

$$\left(e^{-ikx+(k^2-i\beta k)t}u\right)_t = \left(e^{-ikx+(k^2-i\beta k)t}\left[(ik+\beta)u + u_x\right]\right)_x .$$

(ii) Define $\hat{u}(k,t) = \int_0^\infty e^{-ikx}\, u(x,t)\; dx$, which is analytic for $\operatorname{Im} k \leqslant 0$. Determine $\hat{u}(k,t)$ in terms of $\hat{u}(k,0)$ and also the functions f_0, f_1 defined by

$$f_0(\omega,t) = \int_0^t e^{-\omega(t-t')}\, u(0,t')\; dt'\,, \qquad f_1(\omega,t) = \int_0^t e^{-\omega(t-t')}\, u_x(0,t')\; dt'\,.$$

(iii) Show that in the inverse transform expression for $u(x,t)$ the integrals involving f_0, f_1 may be transformed to the contour

$$L = \left\{k \in \mathbb{C} : \operatorname{Re}\,(k^2 - i\beta k) = 0,\ \operatorname{Im} k \geqslant \beta\right\}.$$

By considering $\hat{u}(k',t)$ where $k' = -k + i\beta$ and $k \in L$, show that it is possible to obtain an equation which allows f_1 to be eliminated.

(iv) Obtain an integral expression for the solution of $(*)$ subject to the the initial-boundary value conditions of given $u(x,0)$, $u(0,t)$.

[You need to show that

$$\int_L e^{ikx}\, \hat{u}(k',t)\; dk = 0\,, \quad x > 0\,,$$

by an appropriate closure of the contour which should be justified.]

15D Cosmology

A star has pressure $P(r)$ and mass density $\rho(r)$, where r is the distance from the centre of the star. These quantities are related by the pressure support equation

$$P' = -\frac{Gm\rho}{r^2},$$

where $P' = dP/dr$ and $m(r)$ is the mass within radius r. Use this to derive the virial theorem

$$E_{\rm grav} = -3\,\langle P\rangle\, V\,,$$

where $E_{\rm grav}$ is the total gravitational potential energy and $\langle P\rangle$ the average pressure.

The total kinetic energy of a spherically symmetric star is related to $\langle P\rangle$ by

$$E_{\rm kin} = \alpha\,\langle P\rangle\, V\,,$$

where α is a constant. Use the virial theorem to determine the condition on α for gravitational binding. By considering the relation between pressure and 'internal energy' U for an ideal gas, determine α for the cases of a) an ideal gas of non-relativistic particles, b) an ideal gas of ultra-relativistic particles.

Why does your result imply a maximum mass for any star? Briefly explain what is meant by the Chandrasekhar limit.

A white dwarf is in orbit with a companion star. It slowly accretes matter from the other star until its mass exceeds the Chandrasekhar limit. Briefly explain its subsequent evolution.

16G Logic and Set Theory

Show that ${\aleph_\alpha}^2 = \aleph_\alpha$ for all α.

An infinite cardinal m is called *regular* if it cannot be written as a sum of fewer than m cardinals each of which is smaller than m. Prove that $\aleph_0$ and $\aleph_1$ are regular.

Is $\aleph_2$ regular? Is $\aleph_\omega$ regular? Justify your answers.

17F Graph Theory

(a) Define the *Ramsey number* $R(s)$. Show that for all integers $s \geqslant 2$ the Ramsey number $R(s)$ exists and that $R(s) \leqslant 4^s$.

(b) For any graph G, let $R(G)$ denote the least positive integer n such that in any red-blue colouring of the edges of the complete graph K_n there must be a monochromatic copy of G.

(i) How do we know that $R(G)$ exists for every graph G?

(ii) Let s be a positive integer. Show that, whenever the edge of K_{2s} are red-blue coloured, there must be a monochromatic copy of the complete bipartite graph $K_{1,s}$.

(iii) Suppose s is odd. By exhibiting a suitable colouring of K_{2s-1}, show that $R(K_{1,s}) = 2s$.

(iv) Suppose instead s is even. What is $R(K_{1,s})$? Justify your answer.

18H Galois Theory

Let $\mathbb{F}_q$ be a finite field with q elements and $\overline{\mathbb{F}}_q$ its algebraic closure.

(i) Give a non-zero polynomial $P(X)$ in $\mathbb{F}_q[X_1, \ldots, X_n]$ such that

$$P(\alpha_1, \ldots, \alpha_n) = 0 \qquad \text{for all } \alpha_1, \ldots, \alpha_n \in \mathbb{F}_q.$$

(ii) Show that every irreducible polynomial $P(X)$ of degree $n > 0$ in $\mathbb{F}_q[X]$ can be factored in $\overline{\mathbb{F}}_q[X]$ as $(X-\alpha)(X-\alpha^q)(X-\alpha^{q^2})\cdots(X-\alpha^{q^{n-1}})$ for some $\alpha \in \overline{\mathbb{F}}_q$. What is the splitting field and the Galois group of P over $\mathbb{F}_q$?

(iii) Let n be a positive integer and $\Phi_n(X)$ be the n-th cyclotomic polynomial. Recall that if K is a field of characteristic prime to n, then the set of all roots of Φ_n in K is precisely the set of all primitive n-th roots of unity in K. Using this fact, prove that if p is a prime number not dividing n, then p divides $\Phi_n(x)$ in $\mathbb{Z}$ for some $x \in \mathbb{Z}$ if and only if $p = an + 1$ for some integer a. Write down Φ_n explicitly for three different values of n larger than 2, and give an example of x and p as above for each n.

19F Representation Theory

(i) Let N be a normal subgroup of the finite group G. Without giving detailed proofs, define the process of lifting characters from G/N to G. State also the orthogonality relations for G.

(ii) Let a, b be the following two permutations in S_{12},

$$a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)\,,$$

$$b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)\,,$$

and let $G = \langle a, b\rangle$, a subgroup of S_{12}. Prove that G is a group of order 12 and list the conjugacy classes of G. By identifying a normal subgroup of G of index 4 and lifting irreducible characters, calculate all the linear characters of G. Calculate the complete character table of G. By considering 6th roots of unity, find explicit matrix representations affording the non-linear characters of G.

20G Number Fields

Suppose that m is a square-free positive integer, $m \geqslant 5$, $m \not\equiv 1 \pmod 4$. Show that, if the class number of $K = \mathbb{Q}(\sqrt{-m})$ is prime to 3, then $x^3 = y^2 + m$ has at most two solutions in integers. Assume the m is even.

21H Algebraic Topology

State the path lifting and homotopy lifting lemmas for covering maps. Suppose that X is path connected and locally path connected, that $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are covering maps, and that Y_1 and Y_2 are simply connected. Using the lemmas you have stated, but without assuming the correspondence between covering spaces and subgroups of π_1, prove that Y_1 is homeomorphic to Y_2.

22H Linear Analysis

a) State and prove the Banach–Steinhaus Theorem.

[You may use the Baire Category Theorem without proving it.]

b) Let X be a (complex) normed space and $S \subset X$. Prove that if $\{f(x) : x \in S\}$ is a bounded set in $\mathbb{C}$ for every linear functional $f \in X^*$ then there exists $K \geqslant 0$ such that $\|x\| \leqslant K$ for all $x \in S$.

[You may use here the following consequence of the Hahn–Banach Theorem without proving it: for a given $x \in X$, there exists $f \in X^*$ with $\|f\| = 1$ and $|f(x)| = \|x\|$.]

c) Conclude that if two norms $\|.\|_1$ and $\|.\|_2$ on a (complex) vector space V are not equivalent, there exists a linear functional $f : V \to \mathbb{C}$ which is continuous with respect to one of the two norms, and discontinuous with respect to the other.

23G Riemann Surfaces

Given a lattice $\Lambda \subset \mathbb{C}$, we may define the corresponding Weierstrass $\wp$-function to be the unique even Λ-periodic elliptic function $\wp$ with poles only on Λ and for which $\wp(z) - 1/z^2 \to 0$ as $z \to 0$. For $w \notin \Lambda$, we set

$$f(z) = \det \begin{pmatrix} 1 & 1 & 1 \\ \wp(z) & \wp(w) & \wp(-z-w) \\ \wp'(z) & \wp'(w) & \wp'(-z-w) \end{pmatrix},$$

an elliptic function with periods Λ. By considering the poles of f, show that f has valency at most 4 (i.e. is at most 4 to 1 on a period parallelogram).

If $w \notin \frac{1}{3}\Lambda$, show that f has at least six distinct zeros. If $w \in \frac{1}{3}\Lambda$, show that f has at least four distinct zeros, at least one of which is a multiple zero. Deduce that the meromorphic function f is identically zero.

If z_1, z_2, z_3 are distinct non-lattice points in a period parallelogram such that $z_1 + z_2 + z_3 \in \Lambda$, what can be said about the points $(\wp(z_i), \wp'(z_i)) \in \mathbb{C}^2$ $(i = 1, 2, 3)$?

24G Algebraic Geometry

(i) Let $X = \{(x,y) \in \mathbb{C}^2 \mid x^2 = y^3\}$. Show that X is birational to $\mathbf{A}^1$, but not isomorphic to it.

(ii) Let X be an affine variety. Define the *dimension* of X in terms of the tangent spaces of X.

(iii) Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial, where k is an algebraically closed field of arbitrary characteristic. Show that $\dim Z(f) = n-1$.

[You may assume the Nullstellensatz.]

25H Differential Geometry

(i) State the definition of *smooth manifold with boundary* and define the notion of *boundary.* Show that the boundary ∂X is a manifold (without boundary) with $\dim \partial X = \dim X - 1$.

(ii) Let $0 < a < 1$ and let x_1, x_2, x_3, x_4 denote Euclidean coordinates on $\mathbb{R}^4$. Show that the set

$$X = \{x_1^2 + x_2^2 + x_3^2 - x_4^2 \leqslant a\} \cap \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \cap \{x_1^2 + 2x_2^2 + x_3^2 + x_4^2 = 3/2\}$$

is a manifold with boundary and compute its dimension. You may appeal to standard results concerning regular values of smooth functions.

(iii) Determine if the following statements are true or false, giving reasons:

a. If X and Y are manifolds, $f : X \to Y$ smooth and $Z \subset Y$ a submanifold of codimension r such that f is not transversal to Z, then $f^{-1}(Z)$ is not a submanifold of codimension r in X.

b. If X and Y are manifolds and $f : X \to Y$ is smooth, then the set of regular values of f is open in Y.

c. If X and Y are manifolds and $f : X \to Y$ is smooth then the set of critical points is of measure 0 in X.

26I Probability and Measure

State Carathéodory's extension theorem. Define all terms used in the statement.

Let $\mathcal{A}$ be the ring of finite unions of disjoint bounded intervals of the form

$$A = \bigcup_{i=1}^{m} (a_i, b_i]$$

where $m \in \mathbb{Z}^+$ and $a_1 < b_1 < \ldots < a_m < b_m$. Consider the set function μ defined on $\mathcal{A}$ by

$$\mu(A) = \sum_{i=1}^{m} (b_i - a_i)\,.$$

You may assume that μ is additive. Show that for any decreasing sequence $(B_n : n \in \mathbb{N})$ in $\mathcal{A}$ with empty intersection we have $\mu(B_n) \to 0$ as $n \to \infty$.

Explain how this fact can be used in conjunction with Carathéodory's extension theorem to prove the existence of Lebesgue measure.

27I Applied Probability

(a) Define what it means to say that π is an equilibrium distribution for a Markov chain on a countable state space with Q-matrix $Q = (q_{ij})$, and give an equation which is satisfied by any equilibrium distribution. Comment on the possible non-uniqueness of equilibrium distributions.

(b) State a theorem on convergence to an equilibrium distribution for a continuous-time Markov chain.

A continuous-time Markov chain $(X_t,\ t \geqslant 0)$ has three states 1, 2, 3 and the Q-matrix $Q = (q_{ij})$ is of the form

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1/2 & \lambda_1/2 \\ \lambda_2/2 & -\lambda_2 & \lambda_2/2 \\ \lambda_3/2 & \lambda_3/2 & -\lambda_3 \end{pmatrix},$$

where the rates $\lambda_1, \lambda_2, \lambda_3 \in [0, \infty)$ are not all zero.

[Note that some of the λ_i may be zero, and those cases may need special treatment.]

(c) Find the equilibrium distributions of the Markov chain in question. Specify the cases of uniqueness and non-uniqueness.

(d) Find the limit of the transition matrix $P(t) = \exp(tQ)$ when $t \to \infty$.

(e) Describe the jump chain (Y_n) and its equilibrium distributions. If $\widehat{P}$ is the jump probability matrix, find the limit of $\widehat{P}^n$ as $n \to \infty$.

28J Principles of Statistics

The distribution of a random variable X is obtained from the binomial distribution $\mathcal{B}(n; \Pi)$ by conditioning on $X > 0$; here $\Pi \in (0, 1)$ is an unknown probability parameter and n is known. Show that the distributions of X form an exponential family and identify the natural sufficient statistic T, natural parameter Φ, and cumulant function $k(\phi)$. Using general properties of the cumulant function, compute the mean and variance of X when $\Pi = \pi$. Write down an equation for the maximum likelihood estimate $\widehat{\Pi}$ of Π and explain why, when $\Pi = \pi$, the distribution of $\widehat{\Pi}$ is approximately normal $\mathcal{N}(\pi, \pi(1-\pi)/n)$ for large n.

Suppose we observe $X = 1$. It is suggested that, since the condition $X > 0$ is then automatically satisfied, general principles of inference require that the inference to be drawn should be the same as if the distribution of X had been $\mathcal{B}(n; \Pi)$ and we had observed $X = 1$. Comment briefly on this suggestion.

29I Stochastic Financial Models

What is a Brownian motion? State the reflection principle for Brownian motion.

Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion. Let $M = \max_{0 \leqslant t \leqslant 1} W_t$. Prove

$$\mathbb{P}(M \geqslant x,\ W_1 \leqslant x - y) = \mathbb{P}(M \geqslant x,\ W_1 \geqslant x + y)$$

for all $x, y \geqslant 0$. Hence, show that the random variables M and $|W_1|$ have the same distribution.

Find the density function of the random variable $R = W_1/M$.

30E Partial Differential Equations

(a) Solve by using the method of characteristics

$$x_1 \frac{\partial}{\partial x_1} u + 2\,x_2 \frac{\partial}{\partial x_2} u \;=\; 5\,u\,, \qquad u(x_1,1) \;=\; g(x_1)\,,$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. What is the maximal domain in $\mathbb{R}^2$ in which u is a solution of the Cauchy problem?

(b) Prove that the function

$$u(x,t) \;=\; \begin{cases} 0\,, & x<0\,,\, t>0\,, \\ x/t\,, & 0<x<t\,,\, t>0\,, \\ 1\,, & x>t>0\,, \end{cases}$$

is a weak solution of the Burgers equation

$$\frac{\partial}{\partial t} u + \frac{1}{2}\frac{\partial}{\partial x} u^2 \;=\; 0\,, \qquad x \in \mathbb{R},\, t>0\,, \tag{$*$}$$

with initial data

$$u(x,0) = \begin{cases} 0\,, & x<0\,, \\ 1\,, & x>0\,. \end{cases}$$

(c) Let $u = u(x,t)$, $x \in \mathbb{R}$, $t>0$ be a piecewise C^1-function with a jump discontinuity along the curve

$$\Gamma : x \;=\; s(t)$$

and let u solve the Burgers equation $(*)$ on both sides of Γ. Prove that u is a weak solution of (1) if and only if

$$\dot{s}(t) \;=\; \frac{1}{2}\,(u_l(t) + u_r(t))$$

holds, where $u_l(t)$, $u_r(t)$ are the one-sided limits

$$u_l(t) \;=\; \lim_{x \nearrow s(t)^-} u(x,t)\,, \qquad u_r(t) \;=\; \lim_{x \searrow s(t)^+} u(x,t)\,.$$

[*Hint: Multiply the equation by a test function* $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, *split the integral appropriately and integrate by parts. Consider how the unit normal vector along* Γ *can be expressed in terms of* $\dot{s}$.]

31C Asymptotic Methods

For $\lambda > 0$ let

$$I(\lambda) \;=\; \int_0^b f(x)\,\mathrm{e}^{-\lambda x}\,dx\,, \qquad \text{with} \quad 0 < b < \infty\,.$$

Assume that the function $f(x)$ is continuous on $0 < x \leqslant b$, and that

$$f(x) \;\sim\; x^{\alpha} \sum_{n=0}^{\infty} a_n\, x^{n\beta},$$

as $x \to 0_+$, where $\alpha > -1$ and $\beta > 0$.

(a) Explain briefly why in this case straightforward partial integrations in general cannot be applied for determining the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$.

(b) Derive with proof an asymptotic expansion for $I(\lambda)$ as $\lambda \to \infty$.

(c) For the function

$$B(s,t) = \int_0^1 \; u^{s-1}\,(1-u)^{t-1}\,du\,, \qquad s,t > 0\,,$$

obtain, using the substitution $u = e^{-x}$, the first two terms in an asymptotic expansion as $s \to \infty$. What happens as $t \to \infty$?

[*Hint: The following formula may be useful*

$$\Gamma(y) \;=\; \int_0^{\infty} x^{y-1}\,\mathrm{e}^{-x}\,dt\,, \qquad \text{for} \quad x > 0\,.$$]

32E Integrable Systems

Define a Poisson structure on an open set $U \subset \mathbb{R}^n$ in terms of an anti-symmetric matrix $\omega^{ab} : U \longrightarrow \mathbb{R}$, where $a, b = 1, \cdots, n$. By considering the Poisson brackets of the coordinate functions x^a show that

$$\sum_{d=1}^{n} \left(\omega^{dc} \frac{\partial \omega^{ab}}{\partial x^d} + \omega^{db} \frac{\partial \omega^{ca}}{\partial x^d} + \omega^{da} \frac{\partial \omega^{bc}}{\partial x^d} \right) = 0 \, .$$

Now set $n = 3$ and consider $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc} x^c$, where ε^{abc} is the totally antisymmetric symbol on $\mathbb{R}^3$ with $\varepsilon^{123} = 1$. Find a non–constant function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$\{f, x^a\} = 0 \, , \qquad a = 1, 2, 3 \, .$$

Consider the Hamiltonian

$$H(x^1, x^2, x^3) = \frac{1}{2} \sum_{a,b=1}^{3} M^{ab} x^a x^b \, ,$$

where M^{ab} is a constant symmetric matrix and show that the Hamilton equations of motion with $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc} x^c$ are of the form

$$\dot{x}^a = \sum_{b,\, c=1}^{3} Q^{abc} x^b x^c \, ,$$

where the constants Q^{abc} should be determined in terms of M^{ab}.

33C Principles of Quantum Mechanics

Two states $|j_1\ m_1\rangle_1$, $|j_2\ m_2\rangle_2$, with angular momenta j_1, j_2, are combined to form states $|J\ M\rangle$ with total angular momentum

$$J\ =\ |j_1 - j_2|,\ |j_1 - j_2| + 1\,,\ \ldots\ ,\ j_1 + j_2\,.$$

Write down the state with $J = M = j_1 + j_2$ in terms of the original angular momentum states. Briefly describe how the other combined angular momentum states may be found in terms of the original angular momentum states.

If $j_1 = j_2 = j$, explain why the state with $J = 0$ must be of the form

$$|0\ 0\rangle\ =\ \sum_{m=-j}^{j} \alpha_m |j\ m\rangle_1 |j\ {-m}\rangle_2\,.$$

By considering $J_+|0\ 0\rangle$, determine a relation between α_{m+1} and α_m, hence find α_m .

If the system is in the state $|j\ j\rangle_1 |j\ {-j}\rangle_2$ what is the probability, written in terms of j, of measuring the combined total angular momentum to be zero?

[Standard angular momentum states $|j\ m\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_\pm |j\ m\rangle\ =\ \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m{\pm}1\rangle\,.$$

Units in which $\hbar = 1$ have been used throughout.]

34B Applications of Quantum Mechanics

Give an account of the variational principle for establishing an upper bound on the ground-state energy, E_0, of a particle moving in a potential $V(x)$ in one dimension.

Explain how an upper bound on the energy of the first excited state can be found in the case that $V(x)$ is a symmetric function.

A particle of mass $2m = \hbar^2$ moves in the potential

$$V(x) \;=\; -V_0\, e^{-x^2}\,, \qquad V_0 \;>\; 0\,.$$

Use the trial wavefunction

$$\psi(x) \;=\; e^{-\frac{1}{2}\,ax^2}\,,$$

where a is a positive real parameter, to establish the upper bound $E_0 \leqslant E(a)$ for the energy of the ground state, where

$$E(a) \;=\; \frac{1}{2}\,a - V_0\,\frac{\sqrt{a}}{\sqrt{1+a}}\,.$$

Show that, for $a > 0$, $E(a)$ has one zero and find its position.

Show that the turning points of $E(a)$ are given by

$$(1+a)^3 \;=\; \frac{V_0^2}{a}\,,$$

and deduce that there is one turning point in $a > 0$ for all $V_0 > 0$.

Sketch $E(a)$ for $a > 0$ and hence deduce that $V(x)$ has at least one bound state for all $V_0 > 0$.

For $0 < V_0 \ll 1$ show that

$$-V_0 \;<\; E_0 \;\leqslant\; \epsilon(V_0)\,,$$

where $\epsilon(V_0) = -\,\frac{1}{2}\,{V_0}^2 + \mathrm{O}({V_0}^4)$.

[You may use the result that $\int_{-\infty}^{\infty} e^{-bx^2}\,dx = \sqrt{\frac{\pi}{b}}$ for $b > 0$.]

35B Electrodynamics

The vector potential A^μ is determined by a current density distribution j^μ in the gauge $\partial_\mu A^\mu = 0$ by

$$\Box\, A^\mu \;=\; -\mu_0\, j^\mu\,, \qquad \Box = -\frac{\partial^2}{\partial t^2} + \nabla^2\,,$$

in units where $c = 1$.

Describe how to justify the result

$$A^\mu(\mathbf{x}, t) \;=\; \frac{\mu_0}{4\,\pi} \int d^3x' \, \frac{j^\mu(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}\,, \qquad\qquad t' \;=\; t - |\mathbf{x} - \mathbf{x}'|\,.$$

A plane square loop of thin wire, edge lengths l, has its centre at the origin and lies in the (x, y) plane. For $t < 0$, no current is flowing in the loop, but at $t = 0$ a constant current I is turned on.

Find the vector potential at the point $(0, 0, z)$ as a function of time due to a single edge of the loop.

What is the electric field due to the entire loop at $(0, 0, z)$ as a function of time? Give a careful justification of your answer.

36B General Relativity

Consider a spacetime $\mathcal{M}$ with a metric $g_{ab}(x^c)$ and a corresponding connection $\Gamma^a_{\ bc}$. Write down the differential equation satisfied by a geodesic $x^a(\lambda)$, where λ is an affine parameter.

Show how the requirement that

$$\delta \int g_{ab}(x^c)\, \frac{d}{d\lambda} x^a(\lambda)\, \frac{d}{d\lambda} x^b(\lambda)\; d\lambda \;=\; 0\,,$$

where δ denotes variation of the path, gives the geodesic equation and determines $\Gamma^a_{\ bc}$.

Show that the timelike geodesics for the 2–manifold with line element

$$ds^2 \;=\; t^{-2}\,(dx^2 - dt^2)$$

are given by

$$t^2 \;=\; x^2 + \alpha\, x + \beta\,,$$

where α and β are constants.

37A Fluid Dynamics II

Write down the Navier-Stokes equation for the velocity $\mathbf{u}(\mathbf{x}, t)$ of an incompressible viscous fluid of density ρ and kinematic viscosity ν. Cast the equation into dimensionless form. Define *rectilinear flow*, and explain why the spatial form of any steady rectilinear flow is independent of the Reynolds number.

(i) Such a fluid is contained between two infinitely long plates at $y = 0, y = a$. The lower plate is at rest while the upper plate moves at constant speed U in the x direction. There is an applied pressure gradient $dp/dx = -G\rho\nu$ in the x direction. Determine the flow field.

(ii) Now there is no applied pressure gradient, but baffles are attached to the lower plate at a distance L from each other $(L \gg a)$, lying between the plates so as to prevent any net volume flux in the x direction. Assuming that far from the baffles the flow is essentially rectilinear, determine the flow field and the pressure gradient in the fluid.

38A Waves

Derive the wave equation governing the velocity potential ϕ for linearized sound waves in a compressible inviscid fluid. How is the pressure disturbance related to the velocity potential?

A semi-infinite straight tube of uniform cross-section is aligned along the positive x-axis with its end at $x = -L$. The tube is filled with fluid of density ρ_1 and sound speed c_1 in $-L < x < 0$ and with fluid of density ρ_2 and sound speed c_2 in $x > 0$. A piston at the end of the tube performs small oscillations such that its position is $x = -L + \epsilon\, e^{i\omega t}$, with $\epsilon \ll L$ and $\epsilon\,\omega \ll c_1$, c_2. Show that the complex amplitude of the velocity potential in $x > 0$ is

$$-\epsilon\, c_1 \left(\frac{c_1}{c_2} \cos\frac{\omega L}{c_1} + i\, \frac{\rho_2}{\rho_1} \sin\frac{\omega L}{c_1} \right)^{-1} .$$

Calculate the time-averaged acoustic energy flux in $x > 0$. Comment briefly on the variation of this result with L for the particular case $\rho_2 \ll \rho_1$ and $c_2 = \mathrm{O}(c_1)$.

39A Numerical Analysis

(a) State the Householder–John theorem and explain its relation to the convergence analysis of splitting methods for solving a system of linear equations $Ax = b$ with a positive definite matrix A.

(b) Describe the Jacobi method for solving a system $Ax = b$, and deduce from the above theorem that if A is a symmetric positive definite *tridiagonal* matrix,

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ c_1 & a_2 & c_2 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & c_{n-2} & a_{n-1} & c_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix},$$

then the Jacobi method converges.

[*Hint: At the last step, you may find it useful to consider two vectors* $x = (x_1, x_2, \dots, x_n)$ *and* $y = ((-1)x_1, (-1)^2 x_2, \dots, (-1)^n x_n)$.]

END OF PAPER