UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 4 June, 2009 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **at most six** questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in bundles, marked **A**, **B**, **C**,...,**J** according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

For any integer $x \geqslant 2$, define $\theta(x) = \sum_{p \leqslant x} \log p$, where the sum is taken over all primes $p \leqslant x$. Put $\theta(1) = 0$. By studying the integer

$$\binom{2n}{n},$$

where $n \geqslant 1$ is an integer, prove that

$$\theta(2n) - \theta(n) < 2n \log 2.$$

Deduce that

$$\theta(x) < (4 \log 2)x,$$

for all $x \geqslant 1$.

2F Topics in Analysis

(a) If $f : (0,1) \to \mathbb{R}$ is continuous, prove that there exists a sequence of polynomials P_n such that $P_n \to f$ uniformly on compact subsets of $(0,1)$.

(b) If $f : (0,1) \to \mathbb{R}$ is continuous and bounded, prove that there exists a sequence of polynomials Q_n such that Q_n are uniformly bounded on $(0,1)$ and $Q_n \to f$ uniformly on compact subsets of $(0,1)$.

3F Geometry of Group Actions

Explain why there are discrete subgroups of the Möbius group $\mathbb{P}SL_2(\mathbb{C})$ which abstractly are free groups of rank 2.

4H Coding and Cryptography

Define a binary code of length 15 with information rate 11/15 which will correct single errors. Show that it has the rate stated and give an explicit procedure for identifying the error. Show that the procedure works.

[*Hint: You may wish to imitate the corresponding discussion for a code of length* 7 .]

5I Statistical Modelling

Consider the linear model $Y = X\beta + \varepsilon$, where $\varepsilon \sim N_n(0, \sigma^2 I)$ and X is an $n \times p$ matrix of full rank $p < n$. Suppose that the parameter β is partitioned into k sets as follows: $\beta^\top = (\beta_1^\top \ \cdots \ \beta_k^\top)$. What does it mean for a pair of sets β_i, β_j, $i \neq j$, to be *orthogonal*? What does it mean for all k sets to be *mutually orthogonal*?

In the model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

where $\varepsilon_i \sim N(0, \sigma^2)$ are independent and identically distributed, find necessary and sufficient conditions on $x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}$ for β_0, β_1 and β_2 to be mutually orthogonal.

If β_0, β_1 and β_2 are mutually orthogonal, what consequence does this have for the joint distribution of the corresponding maximum likelihood estimators $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$?

6A Mathematical Biology

Consider an organism moving on a one-dimensional lattice of spacing a, taking steps either to the right or the left at regular time intervals τ. In this random walk there is a slight bias to the right, that is the probabilities of moving to the right and left, α and β, are such that $\alpha - \beta = \epsilon$, where $0 < \epsilon \ll 1$. Write down the appropriate master equation for this process. Show by taking the continuum limit in space and time that $p(x,t)$, the probability that an organism initially at $x = 0$ is at x after time t, obeys

$$\frac{\partial p}{\partial t} + V \frac{\partial p}{\partial x} = D \frac{\partial^2 p}{\partial x^2}.$$

Express the constants V and D in terms of a, τ, α and β.

7E Dynamical Systems

Consider the one-dimensional real map $x_{n+1} = F(x_n) = r x_n^2 (1 - x_n)$, where $r > 0$. Locate the fixed points and explain for what ranges of the parameter r each fixed point exists. For what range of r does F map the open interval $(0, 1)$ into itself?

Determine the location and type of all the bifurcations from the fixed points which occur. Sketch the location of the fixed points in the (r, x) plane, indicating stability.

8B Further Complex Methods

Suppose that the real function $u(x,y)$ satisfies Laplace's equation in the upper half complex z-plane, $z = x + iy$, $x \in \mathbb{R}$, $y > 0$, where

$$u(x,y) \to 0 \quad \text{as} \quad \sqrt{x^2+y^2} \to \infty, \quad u(x,0) = g(x), \quad x \in \mathbb{R}.$$

The function $u(x,y)$ can then be expressed in terms of the Poisson integral

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y g(\xi)}{(x-\xi)^2+y^2} d\xi, \quad x \in \mathbb{R},\ y > 0.$$

By employing the formula

$$f(z) = 2u\left(\frac{z+\bar{a}}{2}, \frac{z-\bar{a}}{2i}\right) - \overline{f(a)},$$

where a is a complex constant with $\operatorname{Im} a > 0$, show that the analytic function whose real part is $u(x,y)$ is given by

$$f(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{g(\xi)}{\xi - z} d\xi + ic, \quad \operatorname{Im} z > 0,$$

where c is a real constant.

9E Classical Dynamics

(a) Show that the principal moments of inertia of a uniform circular cylinder of radius a, length h and mass M about its centre of mass are $I_1 = I_2 = M(a^2/4 + h^2/12)$ and $I_3 = Ma^2/2$, with the x_3 axis being directed along the length of the cylinder.

(b) Euler's equations governing the angular velocity $(\omega_1, \omega_2, \omega_3)$ of an arbitrary rigid body as viewed in the body frame are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3,$$

$$I_2 \frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1$$

and

$$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2.$$

Show that, for the cylinder of part (a), ω_3 is constant. Show further that, when $\omega_3 \neq 0$, the angular momentum vector precesses about the x_3 axis with angular velocity Ω given by

$$\Omega = \left(\frac{3a^2 - h^2}{3a^2 + h^2}\right)\omega_3.$$

10D Cosmology

(a) Write down an expression for the total gravitational potential energy $E_{\rm grav}$ of a spherically symmetric star of outer radius R in terms of its mass density $\rho(r)$ and the total mass $m(r)$ inside a radius r, satisfying the relation $dm/dr = 4\pi r^2\rho(r)$.

An isotropic mass distribution obeys the pressure-support equation,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,,$$

where $P(r)$ is the pressure. Multiply this expression by $4\pi r^3$ and integrate with respect to r to derive the virial theorem relating the kinetic and gravitational energy of the star

$$E_{\rm kin} = -\tfrac{1}{2}E_{\rm grav}\,,$$

where you may assume for a non-relativistic ideal gas that $E_{\rm kin} = \frac{3}{2}\langle P\rangle V$, with $\langle P\rangle$ the average pressure.

(b) Consider a white dwarf supported by electron Fermi degeneracy pressure $P \approx h^2 n^{5/3}/m_{\rm e}$, where $m_{\rm e}$ is the electron mass and n is the number density. Assume a uniform density $\rho(r) = m_{\rm p}n(r) \approx m_{\rm p}\langle n\rangle$, so the total mass of the star is given by $M = (4\pi/3)\langle n\rangle m_{\rm p}R^3$ where $m_{\rm p}$ is the proton mass. Show that the total energy of the white dwarf can be written in the form

$$E_{\rm total} \;=\; E_{\rm kin} + E_{\rm grav} \;=\; \frac{\alpha}{R^2} \;-\; \frac{\beta}{R}\,,$$

where α, β are positive constants which you should specify. Deduce that the white dwarf has a stable radius $R_{\rm WD}$ at which the energy is minimized, that is,

$$R_{\rm WD} \;\sim\; \frac{h^2 M^{-1/3}}{G m_{\rm e} m_{\rm p}^{5/3}}\,.$$

SECTION II

11G Number Theory

Let p be an odd prime. Prove that there is an equal number of quadratic residues and non-residues in the set $\{1, \ldots, p-1\}$.

If n is an integer prime to p, let m_n be an integer such that $nm_n \equiv 1 \bmod p$. Prove that

$$n(n+1) \equiv n^2(1+m_n) \bmod p,$$

and deduce that

$$\sum_{n=1}^{p-1} \left(\frac{n(n+1)}{p}\right) = -1.$$

12F Topics in Analysis

(a) State Runge's theorem on uniform approximation of analytic functions by polynomials.

(b) Let Ω be an unbounded, connected, proper open subset of $\mathbb{C}$. For any given compact set $K \subset \mathbb{C} \setminus \Omega$ and any $\zeta \in \Omega$, show that there exists a sequence of complex polynomials converging uniformly on K to the function $f(z) = (z-\zeta)^{-1}$.

(c) Give an example, with justification, of a connected open subset Ω of $\mathbb{C}$, a compact subset K of $\mathbb{C}\setminus\Omega$ and a point $\zeta \in \Omega$ such that there is no sequence of complex polynomials converging uniformly on K to the function $f(z) = (z-\zeta)^{-1}$.

13A Mathematical Biology

An activator–inhibitor reaction diffusion system in dimensionless form is given by

$$u_t = u_{xx} + \frac{u^2}{v} - bu, \qquad v_t = dv_{xx} + u^2 - v,$$

where b and d are positive constants. Which is the activitor and which the inhibitor? Determine the positive steady states and show, by an examination of the eigenvalues in a linear stability analysis of the spatially uniform situation, that the reaction kinetics is stable if $b < 1$.

Determine the conditions for the steady state to be driven unstable by diffusion. Show that the parameter domain for diffusion–driven instability is given by $0 < b < 1$, $bd > 3 + 2\sqrt{2}$, and sketch the (b, d) parameter space in which diffusion–driven instability occurs. Further show that at the bifurcation to such an instability the critical wave number k_c is given by $k_c^2 = (1+\sqrt{2})/d$.

14E Dynamical Systems

Consider the dynamical system

$$\begin{aligned} \dot{x} &= -ax - 2xy, \\ \dot{y} &= x^2 + y^2 - b, \end{aligned}$$

where $a \geqslant 0$ and $b > 0$.

(i) Find and classify the fixed points. Show that a bifurcation occurs when $4b = a^2 > 0$.

(ii) After shifting coordinates to move the relevant fixed point to the origin, and setting $a = 2\sqrt{b} - \mu$, carry out an extended centre manifold calculation to reduce the two-dimensional system to one of the canonical forms, and hence determine the type of bifurcation that occurs when $4b = a^2 > 0$. Sketch phase portraits in the cases $0 < a^2 < 4b$ and $0 < 4b < a^2$.

(iii) Sketch the phase portrait in the case $a = 0$. Prove that periodic orbits exist if and only if $a = 0$.

15D Cosmology

In the Zel'dovich approximation, particle trajectories in a flat expanding universe are described by $\mathbf{r}(\mathbf{q},t) = a(t)[\mathbf{q}+\mathbf{\Psi}(\mathbf{q},t)]$, where $a(t)$ is the scale factor of the universe, $\mathbf{q}$ is the unperturbed comoving trajectory and $\mathbf{\Psi}$ is the comoving displacement. The particle equation of motion is

$$\ddot{\mathbf{r}} = -\nabla\Phi - \frac{1}{\rho}\nabla P\,,$$

where ρ is the mass density, P is the pressure ($P \ll \rho c^2$) and Φ is the Newtonian potential which satisfies the Poisson equation $\nabla^2\Phi = 4\pi G\rho$.

(i) Show that the fractional density perturbation and the pressure gradient are given by

$$\delta \equiv \frac{\rho-\bar{\rho}}{\bar{\rho}} \approx -\nabla_{\mathbf{q}}\cdot\mathbf{\Psi}\,, \qquad \nabla P \approx -\bar{\rho}\,\frac{c_s^2}{a}\,\nabla_{\mathbf{q}}^2\,\mathbf{\Psi}\,,$$

where $\nabla_{\mathbf{q}}$ has components $\partial/\partial q_i$, $\bar{\rho} = \bar{\rho}(t)$ is the homogeneous background density and $c_s^2 \equiv \partial P/\partial\rho$ is the sound speed. [You may assume that the Jacobian $|\partial r_i/\partial q_j|^{-1} = |a\,\delta_{ij} + a\,\partial\psi_i/\partial q_j|^{-1} \approx a^{-3}(1-\nabla_{\mathbf{q}}\cdot\mathbf{\Psi})$ for $|\mathbf{\Psi}| \ll |\mathbf{q}|$.]

Use this result to integrate the Poisson equation once and obtain then the evolution equation for the comoving displacement:

$$\ddot{\mathbf{\Psi}} + 2\,\frac{\dot{a}}{a}\,\dot{\mathbf{\Psi}} - 4\pi\,G\bar{\rho}\,\mathbf{\Psi} - \frac{c_s^2}{a^2}\,\nabla_{\mathbf{q}}^2\,\mathbf{\Psi} = 0\,,$$

[You may assume that the integral of $\nabla^2\Phi = 4\pi G\,\bar{\rho}$ is $\nabla\Phi = 4\pi\,G\bar{\rho}\mathbf{r}/3$, that $\mathbf{\Psi}$ is irrotational and that the Raychaudhuri equation is $\ddot{a}/a \approx -4\pi\,G\bar{\rho}/3$ for $P \ll \rho c^2$.]

Consider the Fourier expansion $\delta(\mathbf{x},t) = \sum_{\mathbf{k}}\delta_{\mathbf{k}}\exp(i\,\mathbf{k}\cdot\mathbf{x})$ of the density perturbation using the comoving wavenumber $\mathbf{k}$ ($k = |\mathbf{k}|$) and obtain the evolution equation for the mode $\delta_{\mathbf{k}}$:

$$\ddot{\delta}_{\mathbf{k}} + 2\,\frac{\dot{a}}{a}\,\dot{\delta}_{\mathbf{k}} - (4\pi\,G\bar{\rho} - c_s^2\,k^2/a^2)\,\delta_{\mathbf{k}} = 0\,. \qquad (*)$$

(ii) Consider a flat matter-dominated universe with $a(t) = (t/t_0)^{2/3}$ (background density $\bar{\rho} = 1/(6\pi\,Gt^2)$) and with an equation of state $P = \beta\rho^{4/3}$ to show that $(*)$ becomes

$$\ddot{\delta}_{\mathbf{k}} + \frac{4}{3t}\,\dot{\delta}_{\mathbf{k}} - \frac{1}{t^2}\,(\tfrac{2}{3} - \bar{v}_s^2\,k^2)\,\delta_{\mathbf{k}} = 0\,,$$

where the constant $\bar{v}_s^2 \equiv (4\beta/3)(6\pi\,G)^{-1/3}\,t_0^{4/3}$. Seek power law solutions of the form $\delta_{\mathbf{k}} \propto t^\alpha$ to find the growing and decaying modes

$$\delta_{\mathbf{k}} = A_{\mathbf{k}}\,t^{n+} + B_{\mathbf{k}}\,t^{n-} \quad \text{where} \quad n_\pm = -\tfrac{1}{6} \pm \left[(\tfrac{5}{6})^2 - \bar{v}_s^2\,k^2\right]^{1/2}.$$

16G Logic and Set Theory

Let $x \subseteq \alpha$ be a subset of a (von Neumann) ordinal α taken with the induced ordering. Using the recursion theorem or otherwise show that x is order isomorphic to a unique ordinal $\mu(x)$. Suppose that $x \subseteq y \subseteq \alpha$. Show that $\mu(x) \leqslant \mu(y) \leqslant \alpha$.

Suppose that $x_0 \subseteq x_1 \subseteq x_2 \subseteq \cdots$ is an increasing sequence of subsets of α, with x_i an initial segment of x_j whenever $i < j$. Show that $\mu(\bigcup_n x_n) = \bigcup_n \mu(x_n)$. Does this result still hold if the condition on initial segments is omitted? Justify your answer.

Suppose that $x_0 \supseteq x_1 \supseteq x_2 \supseteq \cdots$ is a decreasing sequence of subsets of α. Why is the sequence $\mu(x_n)$ eventually constant? Is it the case that $\mu(\bigcap_n x_n) = \bigcap_n \mu(x_n)$? Justify your answer.

17F Graph Theory

(a) State Brooks' theorem concerning the chromatic number $\chi(G)$ of a graph G. Prove it in the case when G is 3-connected.

[If you wish to assume that G is regular, you should explain why this assumption is justified.]

(b) State Vizing's theorem concerning the edge-chromatic number $\chi'(G)$ of a graph G.

(c) Are the following statements true or false? Justify your answers.

(1) If G is a connected graph on more than two vertices then $\chi(G) \leqslant \chi'(G)$.

(2) For every ordering of the vertices of a graph G, if we colour G using the greedy algorithm (on this ordering) then the number of colours we use is at most $2\chi(G)$.

(3) For every ordering of the edges of a graph G, if we edge-colour G using the greedy algorithm (on this ordering) then the number of colours we use is at most $2\chi'(G)$.

18H Galois Theory

Let $K = \mathbb{F}_p(x)$, the function field in one variable, and let $G = \mathbb{F}_p$. The group G acts as automorphisms of K by $\sigma_a(x) = x + a$. Show that $K^G = \mathbb{F}_p(y)$, where $y = x^p - x$.

[State clearly any theorems you use.]

Is K/K^G a separable extension?

Now let

$$H = \left\{ \begin{pmatrix} d & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_p, d \in \mathbb{F}_p^* \right\}$$

and let H act on K by $\begin{pmatrix} d & a \\ 0 & 1 \end{pmatrix} x = dx + a$. (The group structure on H is given by matrix multiplication.) Compute K^H. Describe your answer in the form $\mathbb{F}_p(z)$ for an explicit $z \in K$.

Is K^G/K^H a Galois extension? Find the minimum polynomial for y over the field K^H.

19F Representation Theory

Let $G = \mathrm{SU}(2)$. Let V_n be the complex vector space of homogeneous polynomials of degree n in two variables z_1, z_2. Define the usual left action of G on V_n and denote by $\rho_n : G \to \mathrm{GL}(V_n)$ the representation induced by this action. Describe the character χ_n afforded by ρ_n.

Quoting carefully any results you need, show that

(i) The representation ρ_n has dimension $n + 1$ and is irreducible for $n \in \mathbb{Z}_{\geqslant 0}$;

(ii) Every finite-dimensional continuous irreducible representation of G is one of the ρ_n;

(iii) V_n is isomorphic to its dual V_n^*.

20G Algebraic Topology

(i) Suppose that (C, d) and (C', d') are chain complexes, and $f, g : C \to C'$ are chain maps. Define what it means for f and g to be *chain homotopic.*

Show that if f and g are chain homotopic, and $f_*, g_* : H_*(C) \to H_*(C')$ are the induced maps, then $f_* = g_*$.

(ii) Define the *Euler characteristic* of a finite chain complex.

Given that one of the sequences below is exact and the others are not, which is the exact one?

$$0 \to \mathbb{Z}^{11} \to \mathbb{Z}^{24} \to \mathbb{Z}^{20} \to \mathbb{Z}^{13} \to \mathbb{Z}^{20} \to \mathbb{Z}^{25} \to \mathbb{Z}^{11} \to 0,$$
$$0 \to \mathbb{Z}^{11} \to \mathbb{Z}^{24} \to \mathbb{Z}^{20} \to \mathbb{Z}^{13} \to \mathbb{Z}^{20} \to \mathbb{Z}^{24} \to \mathbb{Z}^{11} \to 0,$$
$$0 \to \mathbb{Z}^{11} \to \mathbb{Z}^{24} \to \mathbb{Z}^{19} \to \mathbb{Z}^{13} \to \mathbb{Z}^{20} \to \mathbb{Z}^{23} \to \mathbb{Z}^{11} \to 0.$$

Justify your choice.

21H Linear Analysis

(a) State the Arzela–Ascoli theorem, explaining the meaning of all concepts involved.

(b) Prove the Arzela–Ascoli theorem.

(c) Let K be a compact topological space. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in the Banach space $C(K)$ of real-valued continuous functions over K equipped with the supremum norm $\|\cdot\|$. Assume that for every $x \in K$, the sequence $f_n(x)$ is monotone increasing and that $f_n(x) \to f(x)$ for some $f \in C(K)$. Show that $\|f_n - f\| \to 0$ as $n \to \infty$.

22G Riemann Surfaces

(i) Let $f(z) = \sum_{n=1}^{\infty} z^{2^n}$. Show that the unit circle is the natural boundary of the function element $(D(0,1), f)$, where $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

(ii) Let X be a connected Riemann surface and (D, h) a function element on X into $\mathbb{C}$. Define a *germ* of (D, h) at a point $p \in D$. Let $\mathcal{G}$ be the set of all the germs of function elements on X into $\mathbb{C}$. Describe the topology and the complex structure on $\mathcal{G}$, and show that $\mathcal{G}$ is a covering of X (in the sense of complex analysis). Show that there is a one-to-one correspondence between complete holomorphic functions on X into $\mathbb{C}$ and the connected components of $\mathcal{G}$. [You are not required to prove that the topology on $\mathcal{G}$ is second-countable.]

23G Algebraic Geometry

Let V be a smooth projective curve, and let D be an effective divisor on V. Explain how D defines a morphism ϕ_D from V to some projective space. State the necessary and sufficient conditions for ϕ_D to be finite. State the necessary and sufficient conditions for ϕ_D to be an isomorphism onto its image.

Let V have genus 2, and let K be an effective canonical divisor. Show that the morphism ϕ_K is a morphism of degree 2 from V to $\mathbb{P}^1$.

By considering the divisor $K + P_1 + P_2$ for points P_i with $P_1 + P_2 \not\sim K$, show that there exists a birational morphism from V to a singular plane quartic.

[You may assume the Riemann–Roch Theorem.]

24H Differential Geometry

(a) State and prove the *Theorema Egregium.*

(b) Let X be a minimal surface without boundary in $\mathbb{R}^3$ which is closed as a subset of $\mathbb{R}^3$, and assume that X is not contained in a closed ball. Let Π be a plane in $\mathbb{R}^3$ with the property that $D_n \to \infty$ as $n \to \infty$, where for $n = 0, 1, \ldots,$

$$D_n = \inf_{x \in X, d(x,0) \geqslant n} d(x, \Pi).$$

Here $d(x, y)$ denotes the Euclidean distance between x and y and $d(x, \Pi) = \inf_{y \in \Pi} d(x, y)$. Assume moreover that X contains no planar points. Show that X intersects Π.

25J Probability and Measure

State and prove the first and second Borel–Cantelli lemmas.

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent Cauchy random variables. Thus, each X_n is real-valued, with density function

$$f(x) = \frac{1}{\pi(1 + x^2)}.$$

Show that

$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} = c, \quad \text{almost surely,}$$

for some constant c, to be determined.

26J Applied Probability

(a) Define the Poisson process $(N_t, t \geqslant 0)$ with rate $\lambda > 0$, in terms of its holding times. Show that for all times $t \geqslant 0$, N_t has a Poisson distribution, with a parameter which you should specify.

(b) Let X be a random variable with probability density function

$$f(x) = \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}\mathbf{1}_{\{x>0\}}. \qquad (*)$$

Prove that X is distributed as the sum $Y_1 + Y_2 + Y_3$ of three independent exponential random variables of rate λ. Calculate the expectation, variance and moment generating function of X.

Consider a renewal process $(X_t, t \geqslant 0)$ with holding times having density $(*)$. Prove that the renewal function $m(t) = \mathbb{E}(X_t)$ has the form

$$m(t) = \frac{\lambda t}{3} - \frac{1}{3}p_1(t) - \frac{2}{3}p_2(t),$$

where $p_1(t) = \mathbb{P}\big(N_t = 1 \bmod 3\big)$, $p_2(t) = \mathbb{P}\big(N_t = 2 \bmod 3\big)$ and $(N_t, t \geqslant 0)$ is the Poisson process of rate λ.

(c) Consider the delayed renewal process $\big(X_t^{\mathrm{D}}, t \geqslant 0\big)$ with holding times $S_1^{\mathrm{D}}, S_2, S_3, \dots$ where $(S_n, n \geqslant 1)$, are the holding times of $(X_t, t \geqslant 0)$ from (b). Specify the distribution of S_1^{D} for which the delayed process becomes the renewal process in equilibrium.

[You may use theorems from the course provided that you state them clearly.]

27I Principles of Statistics

What is meant by an *equaliser* decision rule? What is meant by an *extended Bayes* rule? Show that a decision rule that is both an equaliser rule and extended Bayes is minimax.

Let $X_1, \dots, X_n$ be independent and identically distributed random variables with the normal distribution $\mathcal{N}(\theta, h^{-1})$, and let $k > 0$. It is desired to estimate θ with loss function $L(\theta, a) = 1 - \exp\{-\frac{1}{2}k(a-\theta)^2\}$.

Suppose the prior distribution is $\theta \sim \mathcal{N}(m_0, h_0^{-1})$. Find the *Bayes act* and the *Bayes loss* posterior to observing $X_1 = x_1, \dots, X_n = x_n$. What is the *Bayes risk* of the Bayes rule with respect to this prior distribution?

Show that the rule that estimates θ by $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ is minimax.

28I Optimization and Control

Two scalar systems have dynamics

$$x_{t+1} = x_t + u_t + \epsilon_t, \qquad y_{t+1} = y_t + w_t + \eta_t,$$

where $\{\epsilon_t\}$ and $\{\eta_t\}$ are independent sequences of independent and identically distributed random variables of mean 0 and variance 1. Let

$$F(x) = \inf_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty}\Big(x_t^2 + u_t^2\Big)(2/3)^t \,\middle|\, x_0 = x\right],$$

where π is a policy in which u_t depends on only $x_0, \ldots, x_t$.

Show that $G(x) = Px^2 + d$ is a solution to the optimality equation satisfied by $F(x)$, for some P and d which you should find.

Find the optimal controls.

State a theorem that justifies $F(x) = G(x)$.

For each of the two cases (a) $\lambda = 0$ and (b) $\lambda = 1$, find controls $\{u_t, w_t\}$ which minimize

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\Big(x_t^2 + 2\lambda x_t y_t + y_t^2 + u_t^2 + w_t^2\Big)(2/3 + \lambda/12)^t \,\middle|\, x_0 = x\,,\ y_0 = y\right].$$

29J Stochastic Financial Models

What is a *Brownian motion*? State the assumptions of the Black–Scholes model of an asset price, and derive the time-0 price of a European call option struck at K, and expiring at T.

Find the time-0 price of a European call option expiring at T, but struck at S_t, where $t \in (0, T)$, and S_t is the price of the underlying asset at time t.

30B Partial Differential Equations

(a) Consider the nonlinear elliptic problem

$$\begin{cases} \Delta u = f(u,x), & x \in \Omega \subseteq \mathbb{R}^d, \\ u = u_D, & x \in \partial\Omega. \end{cases}$$

Let $\frac{\partial f}{\partial u}(y,x) \geqslant 0$ for all $y \in \mathbb{R}$, $x \in \Omega$. Prove that there exists at most one classical solution.

[*Hint: Use the weak maximum principle.*]

(b) Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a radial function. Prove that the Fourier transform of φ is radial too.

(c) Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a radial function. Solve

$$-\Delta u + u = \varphi(x), \quad x \in \mathbb{R}^n$$

by Fourier transformation and prove that u is a radial function.

(d) State the Lax–Milgram lemma and explain its use in proving the existence and uniqueness of a weak solution of

$$-\Delta u + a(x)u = f(x), \; x \in \Omega,$$

$$u = 0 \;\text{ on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^d$ bounded, $0 \leqslant \underline{a} \leqslant a(x) \leqslant \overline{a} < \infty$ for all $x \in \Omega$ and $f \in L^2(\Omega)$.

31A Asymptotic Methods

Consider the contour-integral representation

$$J_0(x) = \mathrm{Re}\,\frac{1}{i\pi}\int_C e^{ix\cosh t}\,dt$$

of the Bessel function J_0 for real x, where C is any contour from $-\infty - \frac{i\pi}{2}$ to $+\infty + \frac{i\pi}{2}$.

Writing $t = u + iv$, give in terms of the real quantities u, v the equation of the steepest-descent contour from $-\infty - \frac{i\pi}{2}$ to $+\infty + \frac{i\pi}{2}$ which passes through $t = 0$.

Deduce the leading term in the asymptotic expansion of $J_0(x)$, valid as $x \to \infty$

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\,\cos\left(x - \frac{\pi}{4}\right).$$

32B Integrable Systems

Consider the partial differential equation

$$\frac{\partial u}{\partial t} = u^n\frac{\partial u}{\partial x} + \frac{\partial^{2k+1}u}{\partial x^{2k+1}}, \qquad (*)$$

where $u = u(x,t)$ and k, n are non-negative integers.

(i) Find a Lie point symmetry of $(*)$ of the form

$$(x, t, u) \longrightarrow (\alpha x, \beta t, \gamma u), \qquad (**)$$

where (α, β, γ) are non-zero constants, and find a vector field generating this symmetry. Find two more vector fields generating Lie point symmetries of $(*)$ which are not of the form $(**)$ and verify that the three vector fields you have found form a Lie algebra.

(ii) Put $(*)$ in a Hamiltonian form.

33C Principles of Quantum Mechanics

(i) Consider two quantum systems with angular momentum states $|\, j\, m\,\rangle$ and $|\, 1\, q\,\rangle$. The eigenstates corresponding to their combined angular momentum can be written as

$$|\, J\, M\,\rangle \;=\; \sum_{q\,m} C^{J\,M}_{q\,m}\, |\, 1\, q\,\rangle |\, j\, m\,\rangle\,,$$

where $C^{J\,M}_{q\,m}$ are Clebsch–Gordan coefficients for addition of angular momenta one and j. What are the possible values of J and how must q, m and M be related for $C^{J\,M}_{q\,m} \neq 0$?

Construct all states $|\, J\, M\,\rangle$ in terms of product states in the case $j = \frac{1}{2}$.

(ii) A general stationary state for an electron in a hydrogen atom $|n\,\ell\,m\rangle$ is specified by the principal quantum number n in addition to the labels ℓ and m corresponding to the total orbital angular momentum and its component in the 3-direction (electron spin is ignored). An oscillating electromagnetic field can induce a transition to a new state $|n'\,\ell'\,m'\rangle$ and, in a suitable approximation, the amplitude for this to occur is proportional to

$$\langle n'\,\ell'\,m'|\,\hat{x}_i\,|n\,\ell\,m\rangle\,,$$

where $\hat{x}_i$ $(i = 1, 2, 3)$ are components of the electron's position. Give clear but concise arguments based on angular momentum which lead to conditions on ℓ, m, ℓ', m' and i for the amplitude to be non-zero.

Explain briefly how parity can be used to obtain an additional selection rule.

[Standard angular momentum states $|\, j\, m\,\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_{\pm}|\, j\, m\,\rangle \;=\; \sqrt{(j{\mp}m)(j{\pm}m{+}1)}\,|\, j\, m{\pm}1\,\rangle, \quad J_3|\, j\, m\,\rangle = m|\, j\, m\,\rangle.$$

You may also assume that $X_{\pm 1} = \frac{1}{\sqrt{2}}(\mp\hat{x}_1 - i\hat{x}_2)$ and $X_0 = \hat{x}_3$ have commutation relations with orbital angular momentum $\mathbf{L}$ given by

$$[L_3, X_q] \;=\; qX_q\;, \qquad [L_{\pm}, X_q] \;=\; \sqrt{(1{\mp}q)(2{\pm}q)}\, X_{q\pm 1}\,.$$

Units in which $\hbar = 1$ are to be used throughout.]

34D Applications of Quantum Mechanics

An electron of charge $-e$ and mass m is subject to a magnetic field of the form $\mathbf{B} = (0, 0, B(y))$, where $B(y)$ is everywhere greater than some positive constant B_0. In a stationary state of energy E, the electron's wavefunction Ψ satisfies

$$-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} + \frac{ie}{\hbar}\mathbf{A}\right)^2 \Psi + \frac{e\hbar}{2m}\mathbf{B}\cdot\boldsymbol{\sigma}\Psi = E\Psi, \qquad (*)$$

where $\mathbf{A}$ is the vector potential and σ_1, σ_2 and σ_3 are the Pauli matrices.

Assume that the electron is in a spin down state and has no momentum along the z-axis. Show that with a suitable choice of gauge, and after separating variables, equation $(*)$ can be reduced to

$$-\frac{d^2\chi}{dy^2} + (k + a(y))^2\,\chi - b(y)\chi = \epsilon\chi, \qquad (**)$$

where χ depends only on y, ϵ is a rescaled energy, and $b(y)$ a rescaled magnetic field strength. What is the relationship between $a(y)$ and $b(y)$?

Show that $(**)$ can be factorized in the form $M^\dagger M\chi = \epsilon\chi$ where

$$M = \frac{d}{dy} + W(y)$$

for some function $W(y)$, and deduce that ϵ is non-negative.

Show that zero energy states exist for all k and are therefore infinitely degenerate.

35D Statistical Physics

Consider an ideal Bose gas in an external potential such that the resulting density of single particle states is given by

$$g(\varepsilon) \;=\; B\,\varepsilon^{7/2},$$

where B is a positive constant.

(i) Derive an expression for the critical temperature for Bose–Einstein condensation of a gas of N of these atoms.

[Recall

$$\frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{z^{-1}e^x \,-\, 1} \;=\; \sum_{\ell=1}^{\infty}\frac{z^\ell}{\ell^n}.$$

]

(ii) What is the internal energy E of the gas in the condensed state as a function of N and T?

(iii) Now consider the high temperature, classical limit instead. How does the internal energy E depend on N and T?

36C Electrodynamics

A particle of charge of q moves along a trajectory $y^a(s)$ in spacetime where s is the proper time on the particle's world-line.

Explain briefly why, in the gauge $\partial_a A^a = 0$, the potential at the spacetime point x is given by

$$A^a(x) = \frac{\mu_0 q}{2\pi}\int ds\;\frac{dy^a}{ds}\;\theta\left(x^0 - y^0(s)\right)\;\delta\Big((x^c - y^c(s))(x^d - y^d(s))\eta_{cd}\Big)\,.$$

Evaluate this integral for a point charge moving relativistically along the z-axis, $x = y = 0$, at constant velocity v so that $z = vt$.

Check your result by starting from the potential of a point charge at rest

$$\mathbf{A} = 0,$$
$$\phi = \frac{\mu_0 q}{4\pi(x^2 + y^2 + z^2)^{1/2}},$$

and making an appropriate Lorentz transformation.

37E Fluid Dynamics II

An axisymmetric incompressible Stokes flow has the Stokes stream function $\Psi(R,\theta)$ in spherical polar coordinates (R,θ,ϕ). Give expressions for the components u_R and u_θ of the flow field in terms of Ψ, and show that

$$\nabla \times \mathbf{u} = \left(0, 0, -\frac{D^2\Psi}{R\sin\theta}\right),$$

where

$$D^2\Psi = \frac{\partial^2\Psi}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right).$$

Write down the equation satisfied by Ψ.

Verify that the Stokes stream function

$$\Psi(R,\theta) = \frac{1}{2}U\sin^2\theta\left(R^2 - \frac{3}{2}aR + \frac{1}{2}\frac{a^3}{R}\right)$$

represents the Stokes flow past a stationary sphere of radius a, when the fluid at large distance from the sphere moves at speed U along the axis of symmetry.

A sphere of radius a moves vertically upwards in the z direction at speed U through fluid of density ρ and dynamic viscosity μ, towards a free surface at $z = 0$. Its distance d from the surface is much greater than a. Assuming that the surface remains flat, show that the conditions of zero vertical velocity and zero tangential stress at $z = 0$ can be approximately met for large d/a by combining the Stokes flow for the sphere with that of an image sphere of the same radius located symmetrically above the free surface. Hence determine the leading-order behaviour of the horizontal flow on the free surface as a function of r, the horizontal distance from the sphere's centre line.

What is the size of the next correction to your answer as a power of a/d? [Detailed calculation is not required.]

[*Hint: For an axisymmetric vector field* $\mathbf{u}$*,*

$$\nabla \times \mathbf{u} = \left(\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(u_\phi\sin\theta),\ -\frac{1}{R}\frac{\partial}{\partial R}(Ru_\phi),\ \frac{1}{R}\frac{\partial}{\partial R}(Ru_\theta) - \frac{1}{R}\frac{\partial u_R}{\partial\theta}\right).\ ]$$

38A Waves

Starting from the equations of motion for an inviscid, incompressible, stratified fluid of density $\rho_0(z)$, where z is the vertical coordinate, derive the dispersion relation

$$\omega^2 = \frac{N^2\left(k^2+\ell^2\right)}{\left(k^2+\ell^2+m^2\right)}$$

for small amplitude internal waves of wavenumber (k, ℓ, m), where N is the constant Brunt–Väisälä frequency (which should be defined), explaining any approximations you make. Describe the wave pattern that would be generated by a small body oscillating about the origin with small amplitude and frequency ω, the fluid being otherwise at rest.

The body continues to oscillate when the fluid has a slowly-varying velocity $[U(z), 0, 0]$, where $U'(z) > 0$. Show that a ray which has wavenumber $(k_0, 0, m_0)$ with $m_0 < 0$ at $z = 0$ will propagate upwards, but cannot go higher than $z = z_c$, where

$$U(z_c) - U(0) = N\left(k_0^2 + m_0^2\right)^{-1/2}.$$

Explain what happens to the disturbance as z approaches z_c.

39B Numerical Analysis

Prove that all Toeplitz tridiagonal $M \times M$ matrices A of the form

$$A = \begin{bmatrix} a & b & & & \\ -b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & -b & a & b \\ & & & -b & a \end{bmatrix}$$

share the same eigenvectors $(\boldsymbol{v}^{(k)})_{k=1}^{M}$, with the components $\boldsymbol{v}_m^{(k)} = i^m \sin \frac{km\pi}{M+1}$, $m = 1, \ldots, M$, where $i = \sqrt{-1}$, and find their eigenvalues.

The advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T,$$

is approximated by the Crank–Nicolson scheme

$$u_m^{n+1} - u_m^n = \frac{1}{4}\mu\left(u_{m+1}^{n+1} - u_{m-1}^{n+1}\right) + \frac{1}{4}\mu\left(u_{m+1}^{n} - u_{m-1}^{n}\right),$$

where $\mu = \frac{\Delta t}{(\Delta x)^2}$, $\Delta x = \frac{1}{M+1}$, and u_m^n is an approximation to $u(m\Delta x, n\Delta t)$. Assuming that $u(0,t) = u(1,t) = 0$, show that the above scheme can be written in the form

$$B\boldsymbol{u}^{n+1} = C\boldsymbol{u}^n, \quad 0 \leqslant n \leqslant T/\Delta t - 1,$$

where $\boldsymbol{u}^n = [u_1^n, \ldots, u_M^n]^T$ and the real matrices B and C should be found. Using matrix analysis, find the range of μ for which the scheme is stable. [Fourier analysis is not acceptable.]

END OF PAPER