UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday, 3 June, 2009 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

State the law of quadratic reciprocity for the Jacobi symbol $\left(\frac{m}{n}\right)$, where m, n are odd positive integers, and prove this law using the reciprocity law for the Legendre symbol.

Compute the Jacobi symbol $\left(\frac{261}{317}\right)$.

2F Topics in Analysis

(a) State *Chebychev's Equal Ripple Criterion.*

(b) Let n be a positive integer, $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ and

$$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

Use Chebychev's Equal Ripple Criterion to prove that

$$\sup_{x\in[-1,1]} |p(x)| \geqslant 2^{1-n}.$$

[You may use without proof that there is a polynomial $T_n(x)$ in x of degree n, with the coefficient of x^n equal to 2^{n-1}, such that $T_n(\cos\theta) = \cos n\theta$ for all $\theta \in \mathbb{R}$.]

3F Geometry of Group Actions

Describe the geodesics in the hyperbolic plane (in a model of your choice).

Let l_1 and l_2 be geodesics in the hyperbolic plane which do not meet either in the plane or at infinity. By considering the action on a suitable third geodesic, or otherwise, prove that the composite $R_{l_1} \circ R_{l_2}$ of the reflections in the two geodesics has infinite order.

4H Coding and Cryptography

Knowing that

$$25 \equiv 2886^2 \mod 3953$$

and that 3953 is the product of two primes p and q, find p and q.

[You should explain your method in sufficient detail to show that it is reasonably general.]

5I Statistical Modelling

What is meant by an *exponential dispersion family*? Show that the family of Poisson distributions with parameter λ is an exponential dispersion family by explicitly identifying the terms in the definition.

Find the corresponding variance function and deduce directly from your calculations expressions for $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$ when $Y \sim \mathrm{Pois}(\lambda)$.

What is the canonical link function in this case?

6A Mathematical Biology

Consider the reaction system

$$A \xrightarrow{k_1} X, \qquad B + X \xrightarrow{k_2} Y, \qquad 2X + Y \xrightarrow{k_3} 3X, \qquad X \xrightarrow{k_4} E,$$

where the ks are the rate constants, and the reactant concentrations of A and B are kept constant. Write down the governing differential equation system for the concentrations of X and Y and nondimensionalise the equations by setting $u = \alpha X$ and $v = \alpha Y$, $\tau = k_4 t$ so that they become

$$\frac{du}{d\tau} = 1 - (b+1)u + au^2 v, \qquad \frac{dv}{d\tau} = bu - au^2 v,$$

by suitable choice of α. Thus find a and b. Determine the positive steady state and show that there is a bifurcation value $b = b_c = 1 + a$ at which the steady state becomes unstable to a Hopf bifurcation. Find the period of the oscillations in the neighbourhood of b_c.

7E Dynamical Systems

For each of the one-dimensional systems

(i) $\dot{x} = \mu^2 - a^2 + 2ax^2 - x^4$,

(ii) $\dot{x} = x(\mu^2 - a^2 + 2ax^2 - x^4)$,

determine the location and stability of all the fixed points. For each system sketch bifurcation diagrams in the (μ, x) plane in each of the two cases $a > 0$ and $a < 0$. Identify and carefully describe all the bifurcation points that occur.

[Detailed calculations are not required, but bifurcation diagrams must be clearly labelled, and the locations of bifurcation points should be given.]

8B Further Complex Methods

The Hilbert transform $\hat{f}$ of a function f is defined by

$$\hat{f}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)}{y-x}\, dy\,,$$

where P denotes the Cauchy principal value.

(i) Compute the Hilbert transform of $(1-\cos t)/t$.

(ii) Solve the following Riemann–Hilbert problem: Find $f^{+}(z)$ and $f^{-}(z)$, which are analytic functions in the upper and lower half z-planes respectively, such that

$$f^{+}(x) - f^{-}(x) = \frac{1-\cos x}{x}\,, \quad x \in \mathbb{R}\,,$$

$$f^{\pm}(z) = O\left(\frac{1}{z}\right), \quad z \to \infty\,, \quad \operatorname{Im} z \neq 0\,.$$

9E Classical Dynamics

A system of three particles of equal mass m moves along the x axis with x_i denoting the x coordinate of particle i. There is an equilibrium configuration for which $x_1 = 0$, $x_2 = a$ and $x_3 = 2a$.

Particles 1 and 2, and particles 2 and 3, are connected by springs with spring constant μ that provide restoring forces when the respective particle separations deviate from their equilibrium values. In addition, particle 1 is connected to the origin by a spring with spring constant $16\mu/3$. The Lagrangian for the system is

$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{\eta}_1^2 + \dot{\eta}_2^2\right) - \frac{\mu}{2}\left(\frac{16}{3}x_1^2 + (\eta_1 - x_1)^2 + (\eta_2 - \eta_1)^2\right),$$

where the generalized coordinates are x_1, $\eta_1 = x_2 - a$ and $\eta_2 = x_3 - 2a$.

Write down the equations of motion. Show that the generalized coordinates can oscillate with a period $P = 2\pi/\omega$, where

$$\omega^2 = \frac{\mu}{3m},$$

and find the form of the corresponding normal mode in this case.

10D Cosmology

(a) The equilibrium distribution for the energy density of a massless neutrino takes the form

$$\epsilon = \frac{4\pi c}{h^3} \int_0^\infty \frac{p^3 dp}{\exp(pc/kT) + 1} \,.$$

Show that this can be expressed in the form $\epsilon = \alpha T^4$, where the constant α need not be evaluated explicitly.

(b) In the early universe, the entropy density s at a temperature T is $s = (8\sigma/3c)\mathcal{N}_S T^3$ where $\mathcal{N}_S$ is the total effective spin degrees of freedom. Briefly explain why $\mathcal{N}_S = \mathcal{N}_* + \mathcal{N}_{SD}$, each term of which consists of two separate components as follows: the contribution from each massless species in equilibrium ($T_i = T$) is

$$\mathcal{N}_* = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i \,,$$

and a similar sum for massless species which have decoupled,

$$\mathcal{N}_{SD} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 \,,$$

where in each case g_i is the degeneracy and T_i is the temperature of the species i.

The three species of neutrinos and antineutrinos decouple from equilibrium at a temperature $T \approx 1$MeV, after which positrons and electrons annihilate at $T \approx 0.5$MeV, leaving photons in equilibrium with a small excess population of electrons. Using entropy considerations, explain why the ratio of the neutrino and photon temperatures today is given by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \,.$$

SECTION II

11F Topics in Analysis

(a) State Brouwer's fixed point theorem in the plane.

(b) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors in $\mathbb{R}^2$ making 120^o angles with one another. Let T be the triangle with vertices given by the points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and let I, J, K be the three sides of T. Prove that the following two statements are equivalent:

(1) There exists no continuous function $f : T \to \partial T$ with $f(I) \subseteq I$, $f(J) \subseteq J$ and $f(K) \subseteq K$.

(2) If A, B, C are closed subsets of $\mathbb{R}^2$ such that $T \subseteq A \cup B \cup C$, $I \subseteq A$, $J \subseteq B$ and $K \subseteq C$, then $A \cap B \cap C \neq \emptyset$.

(c) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be continuous positive functions. Show that the system of equations

$$
\begin{aligned}
(1 - x^2)f^2(x, y) - x^2 g^2(x, y) &= 0 \\
(1 - y^2)g^2(x, y) - y^2 f^2(x, y) &= 0
\end{aligned}
$$

has four distinct solutions on the unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

12H Coding and Cryptography

Describe the construction of the Reed–Miller code $RM(m, d)$. Establish its information rate and minimum weight.

Show that every codeword in $RM(d, d-1)$ has even weight. By considering $\mathbf{x} \wedge \mathbf{y}$ with $\mathbf{x} \in RM(m, r)$ and $\mathbf{y} \in RM(m, m-r-1)$, or otherwise, show that $RM(m, m-r-1) \subseteq RM(m, r)^{\perp}$. Show that, in fact, $RM(m, m-r-1) = RM(m, r)^{\perp}$.

13A Mathematical Biology

Travelling bands of microorganisms, chemotactically directed, move into a food source, consuming it as they go. A model for this is given by

$$b_t = \frac{\partial}{\partial x}\left[Db_x - \frac{b\chi}{a}a_x\right], \qquad a_t = -kb,$$

where $b(x,t)$ and $a(x,t)$ are the bacteria and nutrient respectively and D, χ, and k are positive constants. Look for travelling wave solutions, as functions of $z = x - ct$ where c is the wave speed, with the boundary conditions $b \to 0$ as $|z| \to \infty$, $a \to 0$ as $z \to -\infty$, $a \to 1$ as $z \to \infty$. Hence show that $b(z)$ and $a(z)$ satisfy

$$b' = \frac{b}{cD}\left[\frac{kb\chi}{a} - c^2\right], \qquad a' = \frac{kb}{c},$$

where the prime denotes differentiation with respect to z. Integrating db/da, find an algebraic relationship between $b(z)$ and $a(z)$.

In the special case where $\chi = 2D$ show that

$$a(z) = \left[1 + Ke^{-cz/D}\right]^{-1}, \qquad b(z) = \frac{c^2}{kD}e^{-cz/D}\left[1 + Ke^{-cz/D}\right]^{-2},$$

where K is an arbitrary positive constant which is equivalent to a linear translation; it may be set to 1. Sketch the wave solutions and explain the biological interpretation.

14C Further Complex Methods

Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,$$

$$u(x,0) = xe^{-x}, \quad 0 \leqslant x < \infty,$$

$$u(0,t) = \sin t, \quad t \geqslant 0,$$

where u vanishes sufficiently fast for all t as $x \to \infty$.

(i) Express the solution as an integral (which you should not evaluate) in the complex $k-$plane.

(ii) Explain how to use appropriate contour deformation so that the relevant integrand decays exponentially as $|k| \to \infty$.

15E Classical Dynamics

A symmetric top of unit mass moves under the action of gravity. The Lagrangian is given by

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - gl \cos\theta\,,$$

where the generalized coordinates are the Euler angles (θ, ϕ, ψ), the principal moments of inertia are I_1 and I_3 and the distance from the centre of gravity of the top to the origin is l.

Show that $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ and $p_\phi = I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta$ are constants of the motion. Show further that, when $p_\phi = I_3\omega_3$, with $\omega_3 > 0$, the equation of motion for θ is

$$\frac{d^2\theta}{dt^2} = \frac{gl\sin\theta}{I_1}\left(1 - \frac{I_3^2\omega_3^2}{4I_1 gl\cos^4(\theta/2)}\right).$$

Find the possible equilibrium values of θ in the two cases:

(i) $I_3^2\,\omega_3^2 > 4I_1\,gl$,

(ii) $I_3^2\,\omega_3^2 < 4I_1\,gl$.

By considering linear perturbations in the neighbourhoods of the equilibria in each case, find which are unstable and give expressions for the periods of small oscillations about the stable equilibria.

16G Logic and Set Theory

(i) Give an axiom system and rules of inference for the classical propositional calculus, and explain the notion of *syntactic entailment*. What does it mean to say that a set of propositions is consistent? Let P be a set of primitive propositions and let Φ be a maximal consistent set of propositional formulae in the language based on P. Show that there is a valuation $v : P \to \{T, F\}$ with respect to which all members of Φ are true.
[You should state clearly but need not prove those properties of syntactic entailment which you use.]

(ii) Exhibit a theory T which axiomatizes the collection of groups all of whose non-unit elements have infinite order. Is this theory finitely axiomatizable? Is the theory of groups all of whose elements are of finite order axiomatizable? Justify your answers.

17F Graph Theory

(i) Define the Turán graph $T_r(n)$. State and prove Turán's theorem.

(ii) For each value of n and r with $n > r$, exhibit a graph G on n vertices that has fewer edges than $T_{r-1}(n)$ and yet is maximal K_r–free (meaning that G contains no K_r but the addition of any edge to G produces a K_r). In the case $r = 3$, determine the smallest number of edges that such a G can have.

18H Galois Theory

For each of the following polynomials over $\mathbb{Q}$, determine the splitting field K and the Galois group G.

(1) $x^4 - 2x^2 - 25$.

(2) $x^4 - 2x^2 + 25$.

19F Representation Theory

(i) Let G be a finite group. Show that

(1) If χ is an irreducible character of G then so is its conjugate $\bar{\chi}$.

(2) The product of any two characters of G is again a character of G.

(3) If χ and ψ are irreducible characters of G then

$$\langle \chi\psi, 1_G \rangle = \begin{cases} 1\,, & \text{if } \chi = \bar{\psi}\,, \\ 0\,, & \text{if } \chi \neq \bar{\psi}\,. \end{cases}$$

(ii) If χ is a character of the finite group G, define χ_S and χ_A. For $g \in G$ prove that

$$\chi_S(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right) \quad \text{and} \quad \chi_A(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right).$$

(iii) A certain group of order 24 has precisely seven conjugacy classes with representatives $g_1, \ldots, g_7$; further, G has a character χ with values as follows:

g_i	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\lvert C_G(g_i)\rvert$	24	24	4	6	6	6	6
χ	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2

where $\omega = e^{2\pi i/3}$.

It is given that $g_1^2\,, g_2^2\,, g_3^2\,, g_4^2\,, g_5^2\,, g_6^2\,, g_7^2$ are conjugate to $g_1\,, g_1\,, g_2\,, g_5\,, g_4\,, g_4\,, g_5$ respectively.

Determine χ_S and χ_A, and show that both are irreducible.

20H Number Fields

Suppose that K is a number field of degree $n = r + 2s$, where K has exactly r real embeddings.

(i) Taking for granted the fact that there is a constant C_K such that every integral ideal I of $\mathcal{O}_K$ has a non-zero element x such that $|N(x)| \leqslant C_K\, N(I)$, deduce that the class group of K is finite.

(ii) Compute the class group of $\mathbb{Q}(\sqrt{-21})$, given that you can take

$$C_K = \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n} |D_K|^{1/2},$$

where D_K is the discriminant of K.

(iii) Find all integer solutions of $y^2 = x^3 - 21$.

21G Algebraic Topology

Let $p : X \to Y$ be a connected covering map. Define the notion of a *deck transformation* (also known as *covering transformation*) for p. What does it mean for p to be a *regular* (*normal*) covering map?

If $p^{-1}(y)$ contains n points for each $y \in Y$, we say p is n-to-1. Show that p is regular under either of the following hypotheses:

(1) p is 2-to-1,

(2) $\pi_1(Y)$ is abelian.

Give an example of a 3-to-1 cover of $S^1 \vee S^1$ which is regular, and one which is not regular.

22H Linear Analysis

For $1 \leqslant p < \infty$ and a sequence $x = (x_1, x_2, \dots)$, where $x_j \in \mathbb{C}$ for all $j \geqslant 1$, let $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$.
Let $\ell^p = \{x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \geqslant 1 \text{ and } \|x\|_p < \infty\}$.

(a) Let $p, q > 1$ with $1/p + 1/q = 1$, $x = (x_1, x_2, \dots) \in \ell^p$ and $y = (y_1, y_2, \dots) \in \ell^q$. Prove Hölder's inequality:

$$\sum_{j=1}^{\infty} |x_j||y_j| \leqslant \|x\|_p \|y\|_q .$$

(b) Use Hölder's inequality to prove the triangle inequality (known, in this case, as the Minkowski inequality):

$$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p \qquad \text{for every } x, y \in \ell^p \quad \text{and every } 1 < p < \infty.$$

(c) Let $2 \leqslant p < \infty$ and let K be a closed, convex subset of ℓ^p. Let $x \in \ell^p$ with $x \notin K$. Prove that there exists $y \in K$ such that

$$\|x - y\| = \inf_{z \in K} \|x - z\|.$$

[You may use without proof the fact that for every $2 \leqslant p < \infty$ and for every $x, y \in \ell^p$,

$$\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2^{p-1}\left(\|x\|_p^p + \|y\|_p^p\right) .]$$

23G Riemann Surfaces

(a) Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ be a lattice in $\mathbb{C}$, where the imaginary part of τ is positive. Define the terms *elliptic function* with respect to Λ and *order* of an elliptic function.

Suppose that f is an elliptic function with respect to Λ of order $m > 0$. Show that the derivative f' is also an elliptic function with respect to Λ and that its order n satisfies $m + 1 \leqslant n \leqslant 2m$. Give an example of an elliptic function f with $m = 5$ and $n = 6$, and an example of an elliptic function f with $m = 5$ and $n = 9$.
[Basic results about holomorphic maps may be used without proof, provided these are accurately stated.]

(b) State the monodromy theorem. Using the monodromy theorem, or otherwise, prove that if two tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent then the lattices satisfy $\Lambda_2 = a\Lambda_1$, for some $a \in \mathbb{C} \setminus \{0\}$.
[You may assume that $\mathbb{C}$ is simply connected and every biholomorphic map of $\mathbb{C}$ onto itself is of the form $z \mapsto cz+d$, for some $c, d \in \mathbb{C}$, $c \neq 0$.]

24G Algebraic Geometry

Let V be an irreducible variety over an algebraically closed field k. Define the *tangent space* of V at a point P. Show that for any integer $r \geqslant 0$, the set $\{P \in V \mid \dim T_{V,P} \geqslant r\}$ is a closed subvariety of V.

Assume that k has characteristic different from 2. Let $V = V(I) \subset \mathbb{P}^4$ be the variety given by the ideal $I = (F, G) \subset k[X_0, \ldots, X_4]$, where

$$F = X_1X_2 + X_3X_4, \qquad G = X_0X_1 + X_3^2 + X_4^2.$$

Determine the singular subvariety of V, and compute $\dim T_{V,P}$ at each singular point P. [You may assume that V is irreducible.]

25H Differential Geometry

(a) Let $\alpha : I \to \mathbb{R}^3$ be a smooth regular curve, parametrized by arc length, such that $\alpha''(s) \neq 0$ for all $s \in I$. Define the *Frenet frame* associated to α and derive the *Frenet formulae*, identifying *curvature* and *torsion*.

(b) Let $\alpha, \tilde{\alpha} : I \to \mathbb{R}^3$ be as above such that $\tilde{k}(s) = k(s)$, $\tilde{\tau}(s) = -\tau(s)$, where k, $\tilde{k}$ denote the curvature of α, $\tilde{\alpha}$, respectively, and τ, $\tilde{\tau}$ denote the torsion. Show that there exists a $T \in \mathrm{O}_3$ and $v \in \mathbb{R}^3$ such that

$$\alpha = T \circ \tilde{\alpha} + v.$$

[You may appeal to standard facts about ordinary differential equations provided that they are clearly stated.]

(c) Let $\alpha : I \to \mathbb{R}^2$ be a closed regular plane curve, bounding a region K. Let $A(K)$ denote the area of K, and let $k(s)$ denote the signed curvature at $\alpha(s)$.

Show that there exists a point $s_0 \in I$ such that

$$k(s_0) \leqslant \sqrt{\pi / A(K)}.$$

[You may appeal to any standard theorem provided that it is clearly stated.]

26J Probability and Measure

State Kolmogorov's zero-one law.

State Birkhoff's almost everywhere ergodic theorem and von Neumann's L^p-ergodic theorem.

State the strong law of large numbers for independent and identically distributed integrable random variables, and use the results above to prove it.

27J Applied Probability

Let $X_1, X_2, \ldots,$ be a sequence of independent, identically distributed positive random variables, with a common probability density function $f(x)$, $x > 0$. Call X_n a *record value* if $X_n > \max\{X_1, \ldots, X_{n-1}\}$. Consider the sequence of record values

$$V_0 = 0,\ V_1 = X_1, \ldots, V_n = X_{i_n},$$

where

$$i_n = \min\{i \geqslant 1 :\ X_i > V_{n-1}\},\ n > 1.$$

Define the *record process* $(R_t)_{t\geqslant 0}$ by $R_0 = 0$ and

$$R_t = \max\{n \geqslant 1 :\ V_n < t\}, \quad t > 0.$$

(a) By induction on n, or otherwise, show that the joint probability density function of the random variables $V_1, \ldots, V_n$ is given by:

$$f_{V_1,\ldots,V_n}(x_1, \ldots, x_n) = f(x_1)\frac{f(x_2)}{1 - F(x_1)} \times \ldots \times \frac{f(x_n)}{1 - F(x_{n-1})},$$

where $F(x) = \displaystyle\int_0^x f(y)\mathrm{d}y$ is the cumulative distribution function for $f(x)$.

(b) Prove that the random variable R_t has a Poisson distribution with parameter $\Lambda(t)$ of the form

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s,$$

and determine the 'instantaneous rate' $\lambda(s)$.

[*Hint: You may use the formula*

$$\begin{aligned}\mathbb{P}(R_t = k) &= \mathbb{P}\big(V_k \leqslant t < V_{k+1}\big) \\ &= \int_0^t \cdots \int_0^t \mathbf{1}_{\{t_1 < \ldots < t_k\}} f_{V_1,\ldots,V_k}\big(t_1, \ldots, t_k\big) \\ &\qquad \times \mathbb{P}\big(V_{k+1} > t \big| V_1 = t_1, \ldots, V_k = t_k\big) \prod_{j=1}^{k} \mathrm{d}t_j\,,\end{aligned}$$

for any $k \geqslant 1$.]

28I Principles of Statistics

Suppose that the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ has a distribution over $\mathbb{R}^n$ depending on a real parameter θ, with everywhere positive density function $p(\mathbf{x} \mid \theta)$. Define the *maximum likelihood estimator* $\hat{\theta}$, the *score variable* U, the *observed information* $\hat{\jmath}$ and the *expected (Fisher) information* I for the problem of estimating θ from $\mathbf{X}$.

For the case where the (X_i) are independent and identically distributed, show that, as $n \to \infty$, $I^{-1/2}\, U \xrightarrow{d} \mathcal{N}(0,1)$. [You may assume sufficient conditions to allow interchange of integration over the sample space and differentiation with respect to the parameter.] State the asymptotic distribution of $\hat{\theta}$.

The random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is generated according to the rule

$$X_{i+1} = \theta X_i + E_i,$$

where $X_0 = 1$ and the (E_i) are independent and identically distributed from the standard normal distribution $\mathcal{N}(0,1)$. Write down the likelihood function for θ based on data $\mathbf{x} = (x_1, \ldots, x_n)$, find $\hat{\theta}$ and $\hat{\jmath}$ and show that the pair $(\hat{\theta}, \hat{\jmath})$ forms a *minimal sufficient statistic*.

A Bayesian uses the improper prior density $\pi(\theta) \propto 1$. Show that, in the posterior, $S(\theta - \hat{\theta})$ (where S is a statistic that you should identify) has the same distribution as E_1.

29I Optimization and Control

In the context of stochastic dynamic programming, explain what is meant by an *average-reward optimal policy*.

A player has a fair coin and a six-sided die. At each epoch he may choose either to toss the coin or to roll the die. If he tosses the coin and it shows heads then he adds 1 to his total score, and if it shows tails then he adds 0. If he rolls the die then he adds the number showing. He wins a reward of £1 whenever his total score is divisible by 3.

Suppose the player always tosses the coin. Find his average reward per toss.

Still using the above policy, and given that he starts with a total score of x, let $F_s(x)$ be the expected total reward over the next s epochs. Find the value of

$$\lim_{s \to \infty} \left[F_s(x) - F_s(0) \right].$$

Use the policy improvement algorithm to find a policy that produces a greater average reward than the policy of only tossing the coin.

Find the average-reward optimal policy.

30J Stochastic Financial Models

What is a *martingale*? What is a *stopping time*? State and prove the optional sampling theorem.

Suppose that ξ_i are independent random variables with values in $\{-1, 1\}$ and common distribution $\mathbb{P}(\xi = 1) = p = 1 - q$. Assume that $p > q$. Let S_n be the random walk such that $S_0 = 0$, $S_n = S_{n-1} + \xi_n$ for $n \geqslant 1$. For $z \in (0,1)$, determine the set of values of θ for which the process $M_n = \theta^{S_n} z^n$ is a martingale. Hence derive the probability generating function of the random time

$$\tau_k = \inf\{t : S_t = k\},$$

where k is a positive integer. Hence find the mean of τ_k.

Let $\tau_k' = \inf\{t > \tau_k : S_t = k\}$. Clearly the mean of τ_k' is greater than the mean of τ_k; identify the point in your derivation of the mean of τ_k where the argument fails if τ_k is replaced by τ_k'.

31B Partial Differential Equations

(a) Solve the initial value problem for the Burgers equation

$$u_t + \frac{1}{2}(u^2)_x = 0, \quad x \in \mathbb{R},\, t > 0,$$

$$u(x, t = 0) = u_I(x),$$

where

$$u_I(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Use the method of characteristics. What is the maximal time interval in which this (weak) solution is well defined? What is the regularity of this solution?

(b) Apply the method of characteristics to the Burgers equation subject to the initial condition

$$u_I(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

In $\{(x,t)\,|\,0 < x < t\}$ use the ansatz $u(x,t) = f(\frac{x}{t})$ and determine f.

(c) Using the method of characteristics show that the initial value problem for the Burgers equation has a classical solution defined for all $t > 0$ if u_I is continuously differentiable and

$$\frac{du_I}{dx}(x) > 0$$

for all $x \in \mathbb{R}$.

32B Integrable Systems

Let $L = -\partial_x^2 + u(x,t)$ be a Schrödinger operator and let A be another differential operator which does not contain derivatives with respect to t and such that

$$L_t = [L, A].$$

Show that the eigenvalues of L are independent of t, and deduce that if f is an eigenfunction of L then so is $f_t + Af$. [You may assume that L is self–adjoint.]

Let f be an eigenfunction of L corresponding to an eigenvalue λ which is non-degenerate. Show that there exists a function $\hat{f} = \hat{f}(x,t,\lambda)$ such that

$$L\hat{f} = \lambda\hat{f}, \qquad \hat{f}_t + A\hat{f} = 0. \tag{$*$}$$

Assume

$$A = \partial_x^3 + a_1\partial_x + a_0,$$

where $a_k = a_k(x,t)$, $k = 0,1$ are functions. Show that the system $(*)$ is equivalent to a pair of first order matrix PDEs

$$\partial_x F = UF, \qquad \partial_t F = VF,$$

where $F = (\hat{f}, \partial_x\hat{f})^T$ and U, V are 2×2 matrices which should be determined.

33C Principles of Quantum Mechanics

Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be a set of Hermitian operators obeying

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \text{and} \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1\,, \tag{$*$}$$

where $\mathbf{n}$ is any unit vector. Show that $(*)$ implies

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) \;=\; \mathbf{a}\cdot\mathbf{b} \,+\, i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}\,,$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$. Explain, with reference to the properties $(*)$, how $\boldsymbol{\sigma}$ can be related to the intrinsic angular momentum $\mathbf{S}$ for a particle of spin $\frac{1}{2}$.

Show that the operators $P_\pm = \frac{1}{2}(1 \pm \mathbf{n}\cdot\boldsymbol{\sigma})$ are Hermitian and obey

$$P_\pm^2 = P_\pm\;, \qquad P_+P_- = P_-P_+ = 0\,.$$

Show also how $P_\pm$ can be used to write any state $|\chi\rangle$ as a linear combination of eigenstates of $\mathbf{n}\cdot\boldsymbol{\sigma}$. Use this to deduce that if the system is in a normalised state $|\chi\rangle$ when $\mathbf{n}\cdot\boldsymbol{\sigma}$ is measured, then the results ±1 will be obtained with probabilities

$$||\,P_\pm|\chi\rangle\,||^2 \;=\; \tfrac{1}{2}(\,1 \,\pm\, \langle\chi|\mathbf{n}\cdot\boldsymbol{\sigma}|\chi\rangle\,)\,.$$

If $|\chi\rangle$ is a state corresponding to the system having spin up along a direction defined by a unit vector $\mathbf{m}$, show that a measurement will find the system to have spin up along $\mathbf{n}$ with probability $\frac{1}{2}(1+\mathbf{n}\cdot\mathbf{m})$.

34D Applications of Quantum Mechanics

A particle scatters quantum mechanically off a spherically symmetric potential $V(r)$. In the $l = 0$ sector, and assuming $\hbar^2/2m = 1$, the radial wavefunction $u(r)$ satisfies

$$-\frac{d^2u}{dr^2} + V(r)u = k^2u,$$

and $u(0) = 0$. The asymptotic behaviour of u, for large r, is

$$u(r) \sim C\left(S(k)e^{ikr} - e^{-ikr}\right),$$

where C is a constant. Show that if $S(k)$ is analytically continued to complex k, then

$$S(k)S(-k) = 1 \quad \text{and} \quad S(k)^*S(k^*) = 1.$$

Deduce that for real k, $S(k) = e^{2i\delta_0(k)}$ for some real function $\delta_0(k)$, and that $\delta_0(k) = -\delta_0(-k)$.

For a certain potential,

$$S(k) = \frac{(k+i\lambda)(k+3i\lambda)}{(k-i\lambda)(k-3i\lambda)},$$

where λ is a real, positive constant. Evaluate the scattering length a and the total cross section $4\pi a^2$.

Briefly explain the significance of the zeros of $S(k)$.

35D Statistical Physics

The Van der Waals equation of state for a non-ideal gas is

$$\left(p + \frac{aN^2}{V^2}\right)(V - bN) \;=\; NkT,$$

where a and b are constants.

(i) Briefly explain the physical motivation for differences between the Van der Waals and ideal gas equations of state.

(ii) Find the volume dependence (at constant temperature) of the internal energy E and the heat capacity C_V of a Van der Waals gas.

(iii) A Van der Waals gas is initially at temperature T_1 in an insulated container with volume V_1. A small opening is then made so that the gas can expand freely into an empty container, occupying both the old and new containers. The final result is that the gas now occupies a volume $V_2 > V_1$. Calculate the final temperature T_2 assuming C_V is temperature independent. You may assume the process happens quasistatically.

36D General Relativity

A spacetime has line element

$$ds^2 = -dt^2 + t^{2p_1} dx_1^2 + t^{2p_2} dx_2^2 + t^{2p_3} dx_3^2 \,,$$

where p_1, p_2 and p_3 are constants. Calculate the Christoffel symbols.

Find the constraints on p_1, p_2 and p_3 for this spacetime to be a solution of the vacuum Einstein equations with zero cosmological constant. For which values is the spacetime flat?

Show that it is not possible to have all of p_1, p_2 and p_3 strictly positive, so that if they are all non-zero, the spacetime expands in at least one direction and contracts in at least one direction.

[The Riemann tensor is given in terms of the Christoffel symbols by

$$R^a{}_{bcd} = \Gamma^a_{db,c} - \Gamma^a_{cb,d} + \Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb} \,.\Big]$$

37E Fluid Dynamics II

Show that two-dimensional Stokes flow $\mathbf{u} = (u(r,\phi), v(r,\phi), 0)$ in cylindrical polar coordinates (r,ϕ,z) has a stream function $\psi(r,\phi)$, with $u = r^{-1}\partial\psi/\partial\phi$, $v = -\partial\psi/\partial r$, that satisfies the biharmonic equation

$$\nabla^4\psi = 0.$$

Give, in terms of ψ and/or its derivatives, the boundary conditions satisfied by ψ on an impermeable plane of constant ϕ which is either (a) rigid or (b) stress-free.

A rigid plane passes through the origin and lies along $\phi = -\alpha$. Fluid with viscosity μ is confined in the region $-\alpha < \phi < 0$. A uniform tangential stress S is applied on $\phi = 0$. Show that the resulting flow may be described by a stream function ψ of the form $\psi(r,\phi) = Sr^2 f(\phi)$, where $f(\phi)$ is to be found. Hence show that the radial flow $U(r) = u(r,0)$ on $\phi = 0$ is given by

$$U(r) = \frac{Sr}{\mu}\left(\frac{1-\cos 2\alpha - \alpha\sin 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}\right).$$

By expanding this expression for small α show that U and S have the same sign, provided that α is not too large. Discuss the situation when $\alpha > \alpha_c$, where $\tan 2\alpha_c = 2\alpha_c$.

[*Hint: In plane polar coordinates*

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

and the component $\sigma_{r\phi}$ *of the stress tensor takes the form*

$$\sigma_{r\phi} = \mu\left(r\frac{\partial(v/r)}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \phi}\right). \quad]$$

38A Waves

An elastic solid of density ρ has Lamé moduli λ and μ. From the dynamic equation for the displacement vector $\mathbf{u}$, derive equations satisfied by the dilatational and shear potentials ϕ and $\boldsymbol{\psi}$. Show that two types of plane harmonic wave can propagate in the solid, and explain the relationship between the displacement vector and the propagation direction in each case.

A semi-infinite solid occupies the half-space $y < 0$ and is bounded by a traction-free surface at $y = 0$. A plane P-wave is incident on the plane $y = 0$ with angle of incidence θ. Describe the system of reflected waves, calculate the angles at which they propagate, and show that there is no reflected P-wave if

$$4\sigma(1-\sigma)^{1/2}(\beta-\sigma)^{1/2} = (1-2\sigma)^2,$$

where

$$\sigma = \beta \sin^2\theta \quad \text{and} \quad \beta = \frac{\mu}{\lambda+2\mu}.$$

39B Numerical Analysis

The Poisson equation $\nabla^2 u = f$ in the unit square $\Omega = [0,1] \times [0,1]$, equipped with appropriate boundary conditions on $\partial\Omega$, is discretized with the nine-point formula:

$$\begin{aligned}\Gamma_9[u_{m,n}] := -\frac{10}{3}\,u_{m,n} + \frac{2}{3}\,(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1}) \\ +\frac{1}{6}\,(u_{m+1,n+1} + u_{m+1,n-1} + u_{m-1,n+1} + u_{m-1,n-1}) = h^2 f_{m,n},\end{aligned}$$

where $1 \leqslant m, n \leqslant M$, $u_{m,n} \approx u(mh, nh)$, and (mh, nh) are grid points.

(i) Find the local error of approximation.

(ii) Prove that the error is smaller if f happens to satisfy the Laplace equation $\nabla^2 f = 0$.

(iii) Hence show that the modified nine-point scheme

$$\begin{aligned}\Gamma_9[u_{m,n}] &= h^2 f_{m,n} + \frac{1}{12}h^2 \Gamma_5[f_{m,n}] \\ &:= h^2 f_{m,n} + \frac{1}{12}h^2 (f_{m+1,n} + f_{m-1,n} + f_{m,n+1} + f_{m,n-1} - 4f_{m,n})\end{aligned}$$

has the same smaller error as in (ii).

[*Hint. The nine-point discretization of* $\nabla^2 u$ *can be written as*

$$\Gamma_9[u] = (\Gamma_5 + \tfrac{1}{6}\Delta_x^2\Delta_y^2)\,u = (\Delta_x^2 + \Delta_y^2 + \tfrac{1}{6}\Delta_x^2\Delta_y^2)\,u$$

where $\Gamma_5[u] = (\Delta_x^2 + \Delta_y^2)u$ *is the five-point discretization and*

$$\begin{aligned}\Delta_x^2\,u(x,y) &:= u(x-h,y) - 2u(x,y) + u(x+h,y),\\ \Delta_y^2\,u(x,y) &:= u(x,y-h) - 2u(x,y) + u(x,y+h)\,.\,]\end{aligned}$$

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