UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Friday 4 June 2004 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than **FOUR** *questions.*

The number of marks for each question is the same.

Additional credit will be given for a substantially complete answer.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **1F, 10F** *should be in one bundle and* **3H, 4H** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1F Combinatorics

Write an essay on Ramsey's theorem. You should include the finite and infinite versions, together with some discussion of bounds in the finite case, and give at least one application.

2G Representation Theory

Write an essay on the finite-dimensional representations of SU_2, including a proof of their complete reducibility, and a description of the irreducible representations and the decomposition of their tensor products.

3H Galois Theory

Let M/K be a finite Galois extension of fields. Explain what is meant by the *Galois correspondence* between subfields of M containing K and subgroups of $\mathrm{Gal}(M/K)$. Show that if $K \subset L \subset M$ then $\mathrm{Gal}(M/L)$ is a normal subgroup of $\mathrm{Gal}(M/K)$ if and only if L/K is normal. What is $\mathrm{Gal}(L/K)$ in this case?

Let M be the splitting field of $X^4 - 3$ over $\mathbb{Q}$. Prove that $\mathrm{Gal}(M/\mathbb{Q})$ is isomorphic to the dihedral group of order 8. Hence determine all subfields of M, expressing each in the form $\mathbb{Q}(x)$ for suitable $x \in M$.

4H Differentiable Manifolds

Define what it means for a manifold to be *oriented*, and define a *volume form* on an oriented manifold.

Prove carefully that, for a closed connected oriented manifold of dimension n, $H^n(M) = \mathbb{R}$.

[*You may assume the existence of volume forms on an oriented manifold.*]

If M and N are closed, connected, oriented manifolds of the same dimension, define the *degree* of a map $f : M \to N$.

If f has degree $d > 1$ and $y \in N$, can $f^{-1}(y)$ be

(i) infinite? (ii) a single point? (iii) empty?

Briefly justify your answers.

5G Algebraic Topology

Write down the definition of a covering space and a covering map. State and prove the path lifting property for covering spaces and state, *without proof*, the homotopy lifting property.

Suppose that a group G is a group of homeomorphisms of a space X. Prove that, under conditions to be stated, the quotient map $X \to X/G$ is a covering map and that $\pi_1(X/G)$ is isomorphic to G. Give two examples in which this last result can be used to determine the fundamental group of a space.

6H Number Fields

Let K be a finite extension of $\mathbb{Q}$, and $\mathcal{O}$ the ring of integers of K. Write an essay outlining the proof that every non-zero ideal of $\mathcal{O}$ can be written as a product of non-zero prime ideals, and that this factorisation is unique up to the order of the factors.

7G Hilbert Spaces

Suppose that T is a bounded linear operator on an infinite-dimensional Hilbert space H, and that $\langle T(x), x\rangle$ is real and non-negative for each $x \in H$.

(a) Show that T is Hermitian.

(b) Let $w(T) = \sup\{\langle T(x), x\rangle : \|x\| = 1\}$. Show that

$$\|T(x)\|^2 \leqslant w(T)\langle T(x), x\rangle \quad \text{for each } \; x \in H.$$

(c) Show that $\|T\|$ is an approximate eigenvalue for T.

Suppose in addition that T is compact and injective.

(d) Show that $\|T\|$ is an eigenvalue for T, with finite-dimensional eigenspace.

Explain how this result can be used to diagonalise T.

8H Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2 \neq 0$ and $\omega_1/\omega_2 \notin \mathbb{R}$. By constructing an appropriate family of charts, show that the torus $\mathbb{C}/\Lambda$ is a Riemann surface and that the natural projection $\pi : z \in \mathbb{C} \to z + \Lambda \in \mathbb{C}/\Lambda$ is a holomorphic map.

[*You may assume without proof any known topological properties of* $\mathbb{C}/\Lambda$.]

Let $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ be another lattice in $\mathbb{C}$, with $\omega_1', \omega_2' \neq 0$ and $\omega_1'/\omega_2' \notin \mathbb{R}$. By considering paths from 0 to an arbitrary $z \in \mathbb{C}$, show that if $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is a conformal equivalence then

$$f(z+\Lambda) = (az+b) + \Lambda' \qquad \text{for some } a, b, \in \mathbb{C}, \text{ with } a \neq 0.$$

[*Any form of the Monodromy Theorem and basic results on the lifts of paths may be used without proof, provided that these are correctly stated. You may assume without proof that every injective holomorphic function* $F : \mathbb{C} \to \mathbb{C}$ *is of the form* $F(z) = az + b$*, for some* $a, b \in \mathbb{C}$.]

Give an explicit example of a non-constant holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ that is not a conformal equivalence.

9H Algebraic Curves

Let $F(X, Y, Z)$ be an irreducible homogeneous polynomial of degree n, and write $C = \{p \in \mathbb{P}^2 \mid F(p) = 0\}$ for the curve it defines in $\mathbb{P}^2$. Suppose C is smooth. Show that the degree of its canonical class is $n(n-3)$.

Hence, or otherwise, show that a smooth curve of genus 2 does not embed in $\mathbb{P}^2$.

10F Logic, Computation and Set Theory

Write an essay on recursive functions. Your essay should include a sketch of why every computable function is recursive, and an explanation of the existence of a universal recursive function, as well as brief discussions of the Halting Problem and of the relationship between recursive sets and recursively enumerable sets.

[*You may assume that every recursive function is computable. You do* **not** *need to give proofs that particular functions to do with prime-power decompositions are recursive.*]

11I Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X, X_1, X_2, \ldots$ be random variables. Write an essay in which you discuss the statement: if $X_n \to X$ almost everywhere, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$. You should include accounts of monotone, dominated, and bounded convergence, and of Fatou's lemma.

[*You may assume without proof the following fact. Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space, and let* $f : \Omega \to \mathbb{R}$ *be non-negative with finite integral* $\mu(f)$*. If* $(f_n : n \geqslant 1)$ *are non-negative measurable functions with* $f_n(\omega) \uparrow f(\omega)$ *for all* $\omega \in \Omega$*, then* $\mu(f_n) \to \mu(f)$ *as* $n \to \infty$.]

12I Applied Probability

Consider an $M/G/1$ queue with $\rho = \lambda \mathbb{E} S < 1$. Here λ is the arrival rate and $\mathbb{E}S$ is the mean service time. Prove that in equilibrium, the customer's waiting time W has the moment-generating function $M_W(t) = \mathbb{E}\, e^{tW}$ given by

$$M_W(t) = \frac{(1-\rho)t}{t + \lambda(1 - M_S(t))}$$

where $M_S(t) = \mathbb{E} e^{tS}$ is the moment-generating function of service time S.

[*You may assume that in equilibrium, the* $M/G/1$ *queue size* X *at the time immediately after the customer's departure has the probability generating function*

$$\mathbb{E}\, z^X = \frac{(1-\rho)(1-z)M_S(\lambda(z-1))}{M_S(\lambda(z-1)) - z}, \quad 0 \leqslant z < 1\,.]$$

Deduce that when the service times are exponential of rate μ then

$$M_W(t) = 1 - \rho + \frac{\lambda(1-\rho)}{\mu - \lambda - t}, \quad -\infty < t < \mu - \lambda\,.$$

Further, deduce that W takes value 0 with probability $1 - \rho$ and that

$$\mathbb{P}(W > x | W > 0) = e^{-(\mu - \lambda)x}, \quad x > 0\,.$$

Sketch the graph of $\mathbb{P}(W > x)$ as a function of x.

Now consider the $M/G/1$ queue in the heavy traffic approximation, when the service-time distribution is kept fixed and the arrival rate $\lambda \to 1/\mathbb{E}S$, so that $\rho \to 1$. Assuming that the second moment $\mathbb{E}S^2 < \infty$, check that the limiting distribution of the re-scaled waiting time $\tilde{W}_\lambda = (1 - \lambda \mathbb{E}S)W$ is exponential, with rate $2\mathbb{E}S/\mathbb{E}S^2$.

13J Information Theory

Define a cyclic code of length N.

Show how codewords can be identified with polynomials in such a way that cyclic codes correspond to ideals in the polynomial ring with a suitably chosen multiplication rule.

Prove that any cyclic code $\mathcal{X}$ has a unique *generator*, i.e. a polynomial $c(X)$ of minimum degree, such that the code consists of the multiples of this polynomial. Prove that the rank of the code equals $N - \deg c(X)$, and show that $c(X)$ divides $X^N + 1$.

Let $\mathcal{X}$ be a cyclic code. Set

$$\mathcal{X}^{\perp} = \{y = y_1 \dots y_N : \sum_{i=1}^{N} x_i y_i = 0 \text{ for all } x = x_1 \dots x_N \in \mathcal{X}\}$$

(the dual code). Prove that $\mathcal{X}^{\perp}$ is cyclic and establish how the generators of $\mathcal{X}$ and $\mathcal{X}^{\perp}$ are related to each other.

Show that the repetition and parity codes are cyclic, and determine their generators.

14I Optimization and Control

Consider the deterministic dynamical system

$$\dot{x}_t = Ax_t + Bu_t$$

where A and B are constant matrices, $x_t \in \mathbb{R}^n$, and u_t is the control variable, $u_t \in \mathbb{R}^m$. What does it mean to say that the system is *controllable*?

Let $y_t = e^{-tA}x_t - x_0$. Show that if V_t is the set of possible values for y_t as the control $\{u_s : 0 \leq x \leq t\}$ is allowed to vary, then V_t is a vector space.

Show that each of the following three conditions is equivalent to controllability of the system.

(i) The set $\{v \in \mathbb{R}^n : v^{\top} y_t = 0 \text{ for all } y_t \in V_t\} = \{0\}$.

(ii) The matrix $H(t) = \int_0^t e^{-sA} BB^{\top} e^{-sA^{\top}} ds$ is (strictly) positive definite.

(iii) The matrix $M_n = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]$ has rank n.

Consider the scalar system

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^{n-j} \xi_t = u_t \, ,$$

where $a_0 = 1$. Show that this system is controllable.

15J Principles of Statistics

Suppose that $\theta \in \mathbb{R}^d$ is the parameter of a non-degenerate exponential family. Derive the asymptotic distribution of the maximum-likelihood estimator $\hat{\theta}_n$ of θ based on a sample of size n. [*You may assume that the density is infinitely differentiable with respect to the parameter, and that differentiation with respect to the parameter commutes with integration.*]

16J Stochastic Financial Models

What is *Brownian motion* $(B_t)_{t\geqslant 0}$? Briefly explain how Brownian motion can be considered as a limit of simple random walks. State the *Reflection Principle* for Brownian motion, and use it to derive the distribution of the first passage time $\tau_a \equiv \inf\{t : B_t = a\}$ to some level $a > 0$.

Suppose that $X_t = B_t + ct$, where $c > 0$ is constant. Stating clearly any results to which you appeal, derive the distribution of the first-passage time $\tau_a^{(c)} \equiv \inf\{t : X_t = a\}$ to $a > 0$.

Now let $\sigma_a \equiv \sup\{t : X_t = a\}$, where $a > 0$. Find the density of σ_a.

17B Nonlinear Dynamical Systems

(a) Consider the map $G_1(x) = f(x+a)$, defined on $0 \leqslant x < 1$, where $f(x) = x\,[\text{mod } 1]$, $0 \leqslant f < 1$, and the constant a satisfies $0 \leqslant a < 1$. Give, with reasons, the values of a (if any) for which the map has (i) a fixed point, (ii) a cycle of least period n, (iii) an aperiodic orbit. Does the map exhibit sensitive dependence on initial conditions?

Show (graphically if you wish) that if the map has an n-cycle then it has an infinite number of such cycles. Is this still true if G_1 is replaced by $f(cx + a)$, $0 < c < 1$?

(b) Consider the map

$$G_2(x) = f(x + a + \frac{b}{2\pi}\sin 2\pi x),$$

where $f(x)$ and a are defined as in Part (a), and $b > 0$ is a parameter.

Find the regions of the (a, b) plane for which the map has (i) no fixed points, (ii) exactly two fixed points.

Now consider the possible existence of a 2-cycle of the map G_2 when $b \ll 1$, and suppose the elements of the cycle are X, Y with $X < \frac{1}{2}$. By expanding X, Y, a in powers of b, so that $X = X_0 + bX_1 + b^2X_2 + O(b^3)$, and similarly for Y and a, show that

$$a = \frac{1}{2} + \frac{b^2}{8\pi}\sin 4\pi X_0 + O(b^3).$$

Use this result to sketch the region of the (a, b) plane in which 2-cycles exist. How many distinct cycles are there for each value of a in this region?

18D Partial Differential Equations

(a) State a theorem of local existence, uniqueness and C^1 dependence on the initial data for a solution for an ordinary differential equation. Assuming existence, prove that the solution depends continuously on the initial data.

(b) State a theorem of local existence of a solution for a general quasilinear first–order partial differential equation with data on a smooth non-characteristic hypersurface. Prove this theorem in the linear case assuming the validity of the theorem in part (a); explain in your proof the importance of the non-characteristic condition.

19D Methods of Mathematical Physics

Let $h(t) = i(t + t^2)$. Sketch the path of $\mathrm{Im}(h(t)) = const.$ through the point $t = 0$, and the path of $\mathrm{Im}(h(t)) = const.$ through the point $t = 1$.

By integrating along these paths, show that as $\lambda \to \infty$

$$\int_0^1 t^{-1/2} e^{i\lambda(t+t^2)} dt \sim \frac{c_1}{\lambda^{1/2}} + \frac{c_2 e^{2i\lambda}}{\lambda} ,$$

where the constants c_1 and c_2 are to be computed.

20D Numerical Analysis

Write an essay on the method of conjugate gradients. You should define the method, list its main properties and sketch the relevant proof. You should also prove that (in exact arithmetic) the method terminates in a finite number of steps, briefly mention the connection with Krylov subspaces, and describe the approach of preconditioned conjugate gradients.

21C Electrodynamics

Using Lorentz gauge, $A^a{}_{,a} = 0$, Maxwell's equations for a current distribution J^a can be reduced to $\Box A^a(x) = \mu_0 J^a(x)$. The retarded solution is

$$A^a(x) = \frac{\mu_0}{2\pi} \int d^4y\, \theta(z^0)\delta(z_c z^c) J^a(y),$$

where $z^a = x^a - y^a$. Explain, heuristically, the rôle of the δ-function and Heaviside step function θ in this formula.

The current distribution is produced by a point particle of charge q moving on a world line $r^a(s)$, where s is the particle's proper time, so that

$$J^a(y) = q \int ds\, V^a(s)\delta^{(4)}(y - r(s)),$$

where $V^a = \dot{r}^a(s) = dr^a/ds$. Show that

$$A^a(x) = \frac{\mu_0 q}{2\pi} \int ds\, \theta(X^0)\delta(X_c X^c) V^a(s),$$

where $X^a = x^a - r^a(s)$, and further that, setting $\alpha = X_c V^c$,

$$A^a(x) = \frac{\mu_0 q}{4\pi} \left[\frac{V^a}{\alpha}\right]_{s=s^*},$$

where s^* should be defined. Verify that

$$s^*{}_{,a} = \left[\frac{X_a}{\alpha}\right]_{s=s^*}.$$

Evaluating quantities at $s = s^*$ show that

$$\left[\frac{V^a}{\alpha}\right]_{,b} = \frac{1}{\alpha^2}\left[-V^a V_b + S^a X_b\right],$$

where $S^a = \dot{V}^a + V^a(1 - X_c \dot{V}^c)/\alpha$. Hence verify that $A^a{}_{,a}(x) = 0$ and

$$F_{ab} = \frac{\mu_0 q}{4\pi\alpha^2}\left(S_a X_b - S_b X_a\right).$$

Verify this formula for a stationary point charge at the origin.

[*Hint: If $f(s)$ has simple zeros at s_i, $i = 1, 2, \ldots$ then*

$$\delta(f(s)) = \sum_i \frac{\delta(s_i)}{|f'(s_i)|}.$$

]

22E Foundations of Quantum Mechanics

The states of the hydrogen atom are denoted by $|nlm\rangle$ with $l < n, -l \leq m \leq l$ and associated energy eigenvalue E_n, where

$$E_n = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2} .$$

A hydrogen atom is placed in a weak electric field with interaction Hamiltonian

$$H_1 = -e\mathcal{E}z .$$

a) Derive the necessary perturbation theory to show that to $O(\mathcal{E}^2)$ the change in the energy associated with the state $|100\rangle$ is given by

$$\Delta E_1 = e^2\mathcal{E}^2 \sum_{n=2}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} \frac{\Big|\langle 100|z|nlm\rangle\Big|^2}{E_1 - E_n} . \qquad (*)$$

The wavefunction of the ground state $|100\rangle$ is

$$\psi_{n=1}(\mathbf{r}) = \frac{1}{(\pi a_0^3)^{1/2}}\, e^{-r/a_0} .$$

By replacing E_n, $\forall\ n > 1$, in the denominator of $(*)$ by E_2 show that

$$|\Delta E_1| < \frac{32\pi}{3}\epsilon_0\mathcal{E}^2 a_0^3 .$$

b) Find a matrix whose eigenvalues are the perturbed energies to $O(\mathcal{E})$ for the states $|200\rangle$ and $|210\rangle$. Hence, determine these perturbed energies to $O(\mathcal{E})$ in terms of the matrix elements of z between these states.

[*Hint:*

$$\langle nlm|z|nlm\rangle = 0 \qquad \forall\ n, l, m$$
$$\langle nlm|z|nl'm'\rangle = 0 \qquad \forall\ n, l, l', m, m', \quad m \neq m'$$

]

23E Statistical Physics

Derive the Bose-Einstein expression for the mean number of Bose particles $\bar{n}$ occupying a particular single-particle quantum state of energy ε, when the chemical potential is μ and the temperature is T in energy units.

Why is the chemical potential for a gas of photons given by $\mu = 0$?

Show that, for black-body radiation in a cavity of volume V at temperature T, the mean number of photons in the angular frequency range $(\omega, \omega + d\omega)$ is

$$\frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/T} - 1} \; .$$

Hence, show that the total energy E of the radiation in the cavity is

$$E \;=\; KVT^4 \; ,$$

where K is a constant that need not be evaluated.

Use thermodynamic reasoning to find the entropy S and pressure P of the radiation and verify that

$$E - TS + PV \;=\; 0 \; .$$

Why is this last result to be expected for a gas of photons?

24E Applications of Quantum Mechanics

Describe briefly the variational approach to determining approximate energy eigenvalues for a Hamiltonian H.

Consider a Hamiltonian H and two states $|\psi_1\rangle$, $|\psi_2\rangle$ such that

$$\begin{aligned} \langle\psi_1|H|\psi_1\rangle = \langle\psi_2|H|\psi_2\rangle = \mathcal{E}\,, &\quad \langle\psi_2|H|\psi_1\rangle = \langle\psi_1|H|\psi_2\rangle = \varepsilon\,, \\ \langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1\,, &\quad \langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle = s\,. \end{aligned}$$

Show that, by considering a linear combination $\alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle$, the variational method gives

$$\frac{\mathcal{E}-\varepsilon}{1-s}\,, \qquad \frac{\mathcal{E}+\varepsilon}{1+s}\,,$$

as approximate energy eigenvalues.

Consider the Hamiltonian for an electron in the presence of two protons at $\mathbf{0}$ and $\mathbf{R}$,

$$H = \frac{\mathbf{p}^2}{2m} + \frac{e^2}{4\pi\epsilon_0}\Big(\frac{1}{R} - \frac{1}{|\mathbf{r}|} - \frac{1}{|\mathbf{r}-\mathbf{R}|}\Big)\,, \qquad R = |\mathbf{R}|\,.$$

Let $\psi_0(\mathbf{r}) = e^{-r/a}/(\pi a^3)^{\frac{1}{2}}$ be the ground state hydrogen atom wave function which satisfies

$$\Big(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{4\pi\epsilon_0|\mathbf{r}|}\Big)\psi_0(\mathbf{r}) = E_0\psi_0(\mathbf{r})\,.$$

It is given that

$$\begin{aligned} S &= \int \mathrm{d}^3r\,\psi_0(\mathbf{r})\psi_0(\mathbf{r}-\mathbf{R}) = \Big(1 + \frac{R}{a} + \frac{R^2}{3a^2}\Big)e^{-R/a}\,, \\ U &= \int \mathrm{d}^3r\,\frac{1}{|\mathbf{r}|}\,\psi_0(\mathbf{r})\psi_0(\mathbf{r}-\mathbf{R}) = \frac{1}{a}\Big(1 + \frac{R}{a}\Big)e^{-R/a}\,, \end{aligned}$$

and, for large R, that

$$\int \mathrm{d}^3r\,\frac{1}{|\mathbf{r}-\mathbf{R}|}\,\psi_0(\mathbf{r})^2 - \frac{1}{R} = \mathrm{O}\big(e^{-2R/a}\big)\,.$$

Consider the trial wave function $\alpha\,\psi_0(\mathbf{r}) + \beta\,\psi_0(\mathbf{r}-\mathbf{R})$. Show that the variational estimate for the ground state energy for large R is

$$E(R) = E_0 + \frac{e^2}{4\pi\epsilon_0 R}\big(S - RU\big) + \mathrm{O}\big(e^{-2R/a}\big)\,.$$

Explain why there is an attractive force between the two protons for large R.

25C General Relativity

Starting from the Ricci identity

$$V_{a;b;c} - V_{a;c;b} = V_e R^e{}_{abc},$$

give an expression for the curvature tensor $R^e{}_{abc}$ of the Levi-Civita connection in terms of the Christoffel symbols and their partial derivatives. Using local inertial coordinates, or otherwise, establish that

$$R^e{}_{abc} + R^e{}_{bca} + R^e{}_{cab} = 0. \qquad (*)$$

A vector field with components V^a satisfies

$$V_{a;b} + V_{b;a} = 0. \qquad (**)$$

Show, using equation $(*)$ that

$$V_{a;b;c} = V_e R^e{}_{cba},$$

and hence that

$$V_{a;b}{}^{;b} + R_a{}^c V_c = 0,$$

where R_{ab} is the Ricci tensor. Show that equation $(**)$ may be written as

$$(\partial_c g_{ab})V^c + g_{cb}\partial_a V^c + g_{ac}\partial_b V^c = 0. \qquad (***)$$

If the metric is taken to be the Schwarzschild metric

$$ds^2 = -\left(1 - \tfrac{2M}{r}\right) dt^2 + \left(1 - \tfrac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$

show that $V^a = \delta^a{}_0$ is a solution of $(***)$. Calculate $V^a{}_{;a}$.

Electromagnetism can be described by a vector potential A_a and a Maxwell field tensor F_{ab} satisfying

$$F_{ab} = A_{b;a} - A_{a;b} \qquad \text{and} \qquad F_{ab}{}^{;b} = 0. \qquad (****)$$

The divergence of A_a is arbitrary and we may choose $A_a{}^{;a} = 0$. With this choice show that in a general spacetime

$$A_{a;b}{}^{;b} - R_a{}^c A_c = 0.$$

Hence show that in the Schwarzschild spacetime a tensor field whose only non-trivial components are $F_{tr} = -F_{rt} = Q/r^2$, where Q is a constant, satisfies the field equations $(****)$.

26A Fluid Dynamics II

Write an essay on the Kelvin-Helmholtz instability of a vortex sheet. Your essay should include a detailed linearised analysis, a physical interpretation of the instability, and an informal discussion of nonlinear effects and of the effects of viscosity.

27A Waves in Fluid and Solid Media

A plane shock is moving with speed U into a perfect gas. Ahead of the shock the gas is at rest with pressure p_1 and density ρ_1, while behind the shock the velocity, pressure and density of the gas are u_2, p_2 and ρ_2 respectively. Derive the Rankine-Hugoniot relations across the shock. Show that

$$\frac{\rho_1}{\rho_2} = \frac{2c_1^2 + (\gamma - 1)U^2}{(\gamma + 1)U^2}\,,$$

where $c_1^2 = \gamma p_1/\rho_1$ and γ is the ratio of the specific heats of the gas. Now consider a change of frame such that the shock is stationary and the gas has a component of velocity V parallel to the shock. Deduce that the angle of deflection δ of the flow which is produced by a stationary shock inclined at an angle $\alpha = \tan^{-1}(U/V)$ to an oncoming stream of Mach number $M = (U^2 + V^2)^{\frac{1}{2}}/c_1$ is given by

$$\tan\delta = \frac{2\cot\alpha(M^2\sin\alpha^2 - 1)}{2 + M^2(\gamma + \cos 2\alpha)}\,.$$

[*Note that*

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\,. \quad]$$