UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Thursday 5 June 2003 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)***, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)***. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, ..., J** *according to the letter affixed to each question. (For example,* **5A, 14A** *should be in one bundle and* **1J, 11J** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) Consider the continuous-time Markov chain $(X_t)_{t \geqslant 0}$ with state-space $\{1,2,3,4\}$ and Q-matrix

$$Q = \begin{pmatrix} -2 & 0 & 0 & 2 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 5 & 2 & -8 \end{pmatrix}.$$

Set

$$Y_t = \begin{cases} X_t & \text{if } X_t \in \{1,2,3\} \\ 2 & \text{if } X_t = 4 \end{cases}$$

and

$$Z_t = \begin{cases} X_t & \text{if } X_t \in \{1,2,3\} \\ 1 & \text{if } X_t = 4. \end{cases}$$

Determine which, if any, of the processes $(Y_t)_{t \geqslant 0}$ and $(Z_t)_{t \geqslant 0}$ are Markov chains.

(ii) Find an invariant distribution for the chain $(X_t)_{t \geqslant 0}$ given in Part (i). Suppose $X_0 = 1$. Find, for all $t \geqslant 0$, the probability that $X_t = 1$.

2B Principles of Dynamics

(i) An axisymmetric bowling ball of mass M has the shape of a sphere of radius a. However, it is biased so that the centre of mass is located a distance $a/2$ away from the centre, along the symmetry axis.

The three principal moments of inertia about the centre of mass are (A, A, C). The ball starts out in a stable equilibrium at rest on a perfectly frictionless flat surface with the symmetry axis vertical. The symmetry axis is then tilted through θ_0, the ball is spun about this axis with an angular velocity n, and the ball is released.

Explain why the centre of mass of the ball moves only in the vertical direction during the subsequent motion. Write down the Lagrangian for the ball in terms of the usual Euler angles θ, ϕ and ψ.

(ii) Show that there are three independent constants of the motion. Eliminate two of the angles from the Lagrangian and find the effective Lagrangian for the coordinate θ.

Find the maximum and minimum values of θ in the motion of the ball when the quantity $\frac{C^2 n^2}{AMga}$ is (a) very small and (b) very large.

3G Functional Analysis

(i) Let p be a point of the compact interval $I = [a, b] \subset \mathbb{R}$ and let $\delta_p : C(I) \to \mathbb{R}$ be defined by $\delta_p(f) = f(p)$. Show that

$$\delta_p : (C(I), ||\cdot||_\infty) \to \mathbb{R}$$

is a continuous, linear map but that

$$\delta_p : (C(I), ||\cdot||_1) \to \mathbb{R}$$

is not continuous.

(ii) Consider the space $C^{(n)}(I)$ of n-times continuously differentiable functions on the interval I. Write

$$||f||_\infty^{(n)} = \sum_{k=0}^{n} ||f^{(k)}||_\infty \quad \text{and} \quad ||f||_1^{(n)} = \sum_{r=0}^{n} ||f^{(k)}||_1$$

for $f \in C^{(n)}(I)$. Show that $(C^{(n)}(I), ||\cdot||_\infty^{(n)})$ is a complete normed space. Is the space $(C^{(n)}(I), ||\cdot||_1^{(n)})$ also complete?

Let $f : I \to I$ be an n-times continuously differentiable map and define

$$\mu_f : C^{(n)}(I) \to C^{(n)}(I) \quad \text{by} \quad g \mapsto g \circ f\,.$$

Show that μ_f is a continuous linear map when $C^{(n)}(I)$ is equipped with the norm $||\cdot||_\infty^{(n)}$.

4F Groups, Rings and Fields

(i) Let K be the splitting field of the polynomial $f = X^3 - 2$ over the rationals. Find the Galois group G of $K/\mathbb{Q}$ and describe its action on the roots of f.

(ii) Let K be the splitting field of the polynomial $X^4 + aX^2 + b$ (where $a, b \in \mathbb{Q}$) over the rationals. Assuming that the polynomial is irreducible, prove that the Galois group G of the extension $K/\mathbb{Q}$ is either C_4, or $C_2 \times C_2$, or the dihedral group D_8.

5A Electromagnetism

(i) Given the electric field (in cartesian components)

$$\mathbf{E}(\mathbf{r},t) = \big(0, x/t^2, 0\big)\,,$$

use the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

to find $\mathbf{B}$ subject to the boundary condition that $|\mathbf{B}| \to 0$ as $t \to \infty$.

Let S be the planar rectangular surface in the xy-plane with corners at

$$(0,0,0)\,, \qquad (L,0,0)\,, \qquad (L,a,0)\,, \qquad (0,a,0)$$

where a is a constant and $L = L(t)$ is some function of time. The magnetic flux through S is given by the surface integral

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}.$$

Compute Φ as a function of t.

Let $\mathcal{C}$ be the closed rectangular curve that bounds the surface S, taken anticlockwise in the xy-plane, and let $\mathbf{v}$ be its velocity (which depends, in this case, on the segment of $\mathcal{C}$ being considered). Compute the line integral

$$\oint_{\mathcal{C}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}.$$

Hence verify that

$$\oint_{\mathcal{C}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d\Phi}{dt}\,. \tag{2}$$

(ii) A surface S is bounded by a time-dependent closed curve $\mathcal{C}(t)$ such that in time δt it sweeps out a volume δV. By considering the volume integral

$$\int_{\delta V} \nabla \cdot \mathbf{B}\, d\tau\,,$$

and using the divergence theorem, show that the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ implies that

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_{\mathcal{C}} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

where Φ is the magnetic flux through S as given in Part (i). Hence show, using (1) and Stokes' theorem, that (2) is a consequence of Maxwell's equations.

6D Dynamics of Differential Equations

(i) Define the Poincaré index of a curve $\mathcal{C}$ for a vector field $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$. Explain why the index is uniquely given by the sum of the indices for small curves around each fixed point within $\mathcal{C}$. Write down the indices for a saddle point and for a focus (spiral) or node, and show that the index of a periodic solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has index unity.

A particular system has a periodic orbit containing five fixed points, and two further periodic orbits. Sketch the possible arrangements of these orbits, assuming there are no degeneracies.

(ii) A dynamical system in $\mathbb{R}^2$ depending on a parameter μ has a homoclinic orbit when $\mu = \mu_0$. Explain how to determine the stability of this orbit, and sketch the different behaviours for $\mu < \mu_0$ and $\mu > \mu_0$ in the case that the orbit is stable.

Now consider the system

$$\dot{x} = y \ , \quad \dot{y} = x - x^2 + y(\alpha + \beta x)$$

where α, β are constants. Show that the origin is a saddle point, and that if there is an orbit homoclinic to the origin then α, β are related by

$$\oint y^2(\alpha + \beta x)dt = 0$$

where the integral is taken round the orbit. Evaluate this integral for small α, β by approximating y by its form when $\alpha = \beta = 0$. Hence give conditions on (small) α, β that lead to a stable homoclinic orbit at the origin. [*Note that* $ydt = dx$.]

7H Geometry of Surfaces

(i) Suppose that C is a curve in the Euclidean (ξ, η)-plane and that C is parameterized by its arc length σ. Suppose that S in Euclidean $\mathbb{R}^3$ is the surface of revolution obtained by rotating C about the ξ-axis. Take σ, θ as coordinates on S, where θ is the angle of rotation.

Show that the Riemannian metric on S induced from the Euclidean metric on $\mathbb{R}^3$ is

$$ds^2 = d\sigma^2 + \eta(\sigma)^2 d\theta^2.$$

(ii) For the surface S described in Part (i), let $e_\sigma = \partial/\partial\sigma$ and $e_\theta = \partial/\partial\theta$. Show that, along any geodesic γ on S, the quantity $g(\dot{\gamma}, e_\theta)$ is constant. Here g is the metric tensor on S.

[*You may wish to compute* $[X, e_\theta] = Xe_\theta - e_\theta X$ *for any vector field* $X = Ae_\sigma + Be_\theta$, *where* A, B *are functions of* σ, θ. *Then use symmetry to compute* $D_{\dot{\gamma}}(g(\dot{\gamma}, e_\theta))$, *which is the rate of change of* $g(\dot{\gamma}, e_\theta)$ *along* γ.]

8H Logic, Computation and Set Theory

(i) What does it mean for a function from $\mathbb{N}^k$ to $\mathbb{N}$ to be *recursive*? Write down a function that is not recursive. You should include a proof that your example is not recursive.

(ii) What does it mean for a subset of $\mathbb{N}^k$ to be *recursive*, and what does it mean for it to be *recursively enumerable*? Give, with proof, an example of a set that is recursively enumerable but not recursive. Prove that a set is recursive if and only if both it and its complement are recursively enumerable. If a set is recursively enumerable, must its complement be recursively enumerable?

[*You may assume the existence of any universal recursive functions or universal register machine programs that you wish.*]

9G Number Theory

(i) Let $x \geqslant 2$ be a real number and let $P(x) = \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^{-1}$, where the product is taken over all primes $p \leqslant x$. Prove that $P(x) > \log x$.

(ii) Define the continued fraction of any positive irrational real number x. Illustrate your definition by computing the continued fraction of $1 + \sqrt{3}$.

Suppose that a, b, c are positive integers with $b = ac$ and that x has the periodic continued fraction $[b, a, b, a, \ldots]$. Prove that $x = \frac{1}{2}(b + \sqrt{b^2 + 4c})$.

10I Algorithms and Networks

(i) Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h(x) = b, \quad x \in X, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ and $b \in \mathbb{R}^m$. State and prove the Lagrangian sufficiency theorem.

In each of the following cases, where $n = 2$, $m = 1$ and $X = \{(x, y) : x, y \geqslant 0\}$, determine whether the Lagrangian sufficiency theorem can be applied to solve the problem:

(a) $f(x,y) = -x$, $h(x,y) = x^2 + y^2$, $b = 1$;

(b) $f(x,y) = e^{-xy}$, $h(x) = x$, $b = 0$.

(ii) Consider the problem in $\mathbb{R}^n$

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T Q x + c^T x \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where Q is a positive-definite symmetric $n \times n$ matrix, A is an $m \times n$ matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Explain how to reduce this problem to the solution of simultaneous linear equations.

Consider now the problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T Q x + c^T x \\ \text{subject to} \quad & Ax \geqslant b\,. \end{aligned}$$

Describe the active set method for its solution.

Consider the problem

$$\begin{aligned} \text{minimize} \quad & (x-a)^2 + (y-b)^2 + xy \\ \text{subject to} \quad & 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1 \end{aligned}$$

where $a, b \in \mathbb{R}$. Draw a diagram partitioning the (a,b)-plane into regions according to which constraints are active at the minimum.

11J Stochastic Financial Models

(i) What does it mean to say that the process $(W_t)_{t\geqslant 0}$ is a *Brownian motion?* What does it mean to say that the process $(M_t)_{t\geqslant 0}$ is a *martingale?*

Suppose that $(W_t)_{t\geqslant 0}$ is a Brownian motion and the process $(X_t)_{t\geqslant 0}$ is given in terms of W as

$$X_t = x_0 + \sigma W_t + \mu t$$

for constants σ, μ. For what values of θ is the process

$$M_t = \exp(\theta X_t - \lambda t)$$

a martingale? (Here, λ is a positive constant.)

(ii) In a standard Black–Scholes model, the price at time t of a share is represented as $S_t = \exp(X_t)$. You hold a perpetual American put option on this share, with strike K; you may exercise at any stopping time τ, and upon exercise you receive $\max\{0, K - S_\tau\}$. Let $0 < a < \log K$. Suppose you plan to use the exercise policy: 'Exercise as soon as the price falls to e^a or lower.' Calculate what the option would be worth if you were to follow this policy. (Assume that the riskless rate of interest is constant and equal to $r > 0$.) For what choice of a is this value maximised?

12I Principles of Statistics

(i) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with the exponential density $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$.

Obtain the maximum likelihood estimator $\hat{\theta}$ of θ. What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

What is the minimum variance unbiased estimator of θ? Justify your answer carefully.

(ii) Explain briefly what is meant by the *profile log-likelihood* for a scalar parameter of interest γ, in the presence of a nuisance parameter ξ. Describe how you would test a null hypothesis of the form $H_0 : \gamma = \gamma_0$ using the profile log-likelihood ratio statistic.

In a reliability study, lifetimes $T_1, \ldots, T_n$ are independent and exponentially distributed, with means of the form $E(T_i) = \exp(\beta + \xi z_i)$ where β, ξ are unknown and $z_1, \ldots, z_n$ are known constants. Inference is required for the mean lifetime, $\exp(\beta + \xi z_0)$, for covariate value z_0.

Find, as explicitly as possible, the profile log-likelihood for $\gamma \equiv \beta + \xi z_0$, with nuisance parameter ξ.

Show that, under $H_0 : \gamma = \gamma_0$, the profile log-likelihood ratio statistic has a distribution which does not depend on the value of ξ. How might the parametric bootstrap be used to obtain a test of H_0 of exact size α?

[*Hint: if Y is exponentially distributed with mean* 1, *then μY is exponentially distributed with mean μ.*]

13C Foundations of Quantum Mechanics

(i) What are the commutation relations satisfied by the components of an angular momentum vector $\mathbf{J}$? State the possible eigenvalues of the component J_3 when $\mathbf{J}^2$ has eigenvalue $j(j+1)\hbar^2$.

Describe how the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are used to construct the components of the angular momentum vector $\mathbf{S}$ for a spin $\frac{1}{2}$ system. Show that they obey the required commutation relations.

Show that S_1, S_2 and S_3 each have eigenvalues $\pm\frac{1}{2}\hbar$. Verify that $\mathbf{S}^2$ has eigenvalue $\frac{3}{4}\hbar^2$.

(ii) Let $\mathbf{J}$ and $|jm\rangle$ denote the standard operators and state vectors of angular momentum theory. Assume units where $\hbar = 1$. Consider the operator

$$U(\theta) = e^{-i\theta J_2}\,.$$

Show that

$$\begin{aligned} U(\theta)J_1U(\theta)^{-1} &= \cos\theta J_1 - \sin\theta J_3 \\ U(\theta)J_3U(\theta)^{-1} &= \sin\theta J_1 + \cos\theta J_3\,. \end{aligned}$$

Show that the state vectors $U\left(\frac{\pi}{2}\right)|jm\rangle$ are eigenvectors of J_1. Suppose that J_1 is measured for a system in the state $|jm\rangle$; show that the probability that the result is m' equals

$$|\langle jm'|e^{i\frac{\pi}{2}J_2}|jm\rangle|^2\,.$$

Consider the case $j = m = \frac{1}{2}$. Evaluate the probability that the measurement of J_1 will result in $m' = -\frac{1}{2}$.

14A Statistical Physics and Cosmology

(i) The pressure $P(r)$ and mass density $\rho(r)$, at distance r from the centre of a spherically-symmetric star, obey the pressure-support equation

$$P' = -\frac{Gm\rho}{r^2}$$

where $m' = 4\pi r^2 \rho(r)$, and the prime indicates differentiation with respect to r. Let V be the total volume of the star, and $\langle P \rangle$ its average pressure. Use the pressure-support equation to derive the "virial theorem"

$$\langle P \rangle V = -\frac{1}{3} E_{grav}$$

where E_{grav} is the total gravitational potential energy [*Hint: multiply by* $4\pi r^3$]. If a star is assumed to be a self-gravitating ball of a non-relativistic ideal gas then it can be shown that

$$\langle P \rangle V = \frac{2}{3} E_{kin}$$

where E_{kin} is the total kinetic energy. Use this result to show that the total energy $U = E_{grav} + E_{kin}$ is negative. When nuclear reactions have converted the hydrogen in a star's core to helium the core contracts until the helium is converted to heavier elements, thereby increasing the total energy U of the star. Explain briefly why this converts the star into a "Red Giant".

(ii) Write down the first law of thermodynamics for the change in energy E of a system at temperature T, pressure P and chemical potential μ as a result of small changes in the entropy S, volume V and particle number N. Use this to show that

$$P = -\left(\frac{\partial E}{\partial V}\right)_{N,S}.$$

The microcanonical ensemble is the set of all accessible microstates of a system at fixed E, V, N. Define the canonical and grand-canonical ensembles. Why are the properties of a macroscopic system independent of the choice of thermodynamic ensemble?

The Gibbs "grand potential" $\mathcal{G}(T, V, \mu)$ can be defined as

$$\mathcal{G} = E - TS - \mu N.$$

Use the first law to find expressions for S, P, N as partial derivatives of $\mathcal{G}$. A system with variable particle number n has non-degenerate energy eigenstates labeled by $r^{(n)}$, for each n, with energy eigenvalues $E_r^{(n)}$. If the system is in equilibrium at temperature T and chemical potential μ then the probability $p(r^{(n)})$ that it will be found in a particular n-particle state $r^{(n)}$ is given by the Gibbs probability distribution

$$p(r^{(n)}) = \mathcal{Z}^{-1}\, e^{\left(\mu n - E_r^{(n)}\right)/kT}$$

where k is Boltzmann's constant. Deduce an expression for the normalization factor $\mathcal{Z}$ as a function of μ and $\beta = 1/kT$, and hence find expressions for the partial derivatives

$$\frac{\partial \log \mathcal{Z}}{\partial \mu}, \qquad \frac{\partial \log \mathcal{Z}}{\partial \beta}$$

in terms of N, E, μ, β.

Why does $\mathcal{Z}$ also depend on the volume V? Given that a change in V at fixed N, S leaves unchanged the Gibbs probability distribution, deduce that

$$\left(\frac{\partial \log \mathcal{Z}}{\partial V}\right)_{\mu,\beta} = \beta P.$$

Use your results to show that

$$\mathcal{G} = -kT \log\left(\mathcal{Z}/\mathcal{Z}_0\right)$$

for some constant $\mathcal{Z}_0$.

15C Symmetries and Groups in Physics

(i) Given that the character of an $SU(2)$ transformation in the $(2l+1)$-dimensional irreducible representation d_l is given by

$$\chi_l(\theta) = \frac{\sin(l+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},$$

show how the direct product representation $d_{l_1} \otimes d_{l_2}$ decomposes into irreducible $SU(2)$ representations.

(ii) Find the decomposition of the direct product representation $3 \otimes \overline{3}$ of $SU(3)$ into irreducible $SU(3)$ representations.

Mesons consist of one quark and one antiquark. The scalar Meson Octet consists of the following particles: $K^{\pm}$ ($Y = \pm 1$, $I_3 = \pm\frac{1}{2}$), K^0 ($Y = 1$, $I_3 = -\frac{1}{2}$), $\overline{K}^0$ ($Y = -1$, $I_3 = \frac{1}{2}$), $\pi^{\pm}$ ($Y = 0$, $I_3 = \pm 1$), π^0 ($Y = 0$, $I_3 = 0$) and η ($Y = 0$, $I_3 = 0$).

Use the direct product representation $3 \otimes \overline{3}$ of $SU(3)$ to identify the quark-type of the particles in the scalar Meson Octet. Deduce the quark-type of the $SU(3)$ singlet state η' contained in $3 \otimes \overline{3}$.

16E Transport Processes

(i) When a solid crystal grows into a supercooled infinite melt, latent heat must be removed from the interface by diffusion into the melt. Write down the equation and boundary conditions satisfied by the temperature $\theta(\boldsymbol{x}, t)$ in the melt, where $\boldsymbol{x}$ is position and t time, in terms of the following material properties: solid density ρ_s, specific heat capacity C_p, coefficient of latent heat per unit mass L, thermal conductivity k, melting temperature θ_m. You may assume that the densities of the melt and the solid are the same and that temperature in the melt far from the interface is $\theta_m - \Delta\theta$, where $\Delta\theta$ is a positive constant.

A spherical crystal of radius $a(t)$ grows into such a melt with $a(0) = 0$. Use dimensional analysis to show that $a(t)$ is proportional to $t^{1/2}$.

(ii) Show that the above problem should have a similarity solution of the form

$$\theta = \theta_m - \Delta\theta(1 - F(\xi)),$$

where $\xi = r(\kappa t)^{-1/2}$, r is the radial coordinate in spherical polars and $\kappa = k/\rho_s C_p$ is the thermal diffusivity. Recalling that, for spherically symmetric $\theta, \nabla^2\theta = \frac{1}{r^2}(r^2\theta_r)_r$, write down the equation and boundary conditions to be satisfied by $F(\xi)$. Hence show that the radius of the crystal is given by $a(t) = \lambda(\kappa t)^{1/2}$, where λ satisfies the equation

$$\int_\lambda^\infty \frac{e^{-\frac{1}{4}u^2}}{u^2} du = \frac{2}{S\lambda^3} e^{-\frac{1}{4}\lambda^2}$$

and $S = L/C_p\Delta\theta$.

Integrate the left hand side of this equation by parts, to give

$$\frac{\sqrt{\pi}}{2}\lambda e^{\frac{1}{4}\lambda^2} \operatorname{erfc}\left(\frac{1}{2}\lambda\right) = 1 - \frac{2}{S\lambda^2}.$$

Hence show that a solution with λ small must have $\lambda \approx (2/S)^{\frac{1}{2}}$, which is self-consistent if S is large.

17B Mathematical Methods

(i) Explain what is meant by the assertion: "the series $\sum_0^\infty b_n x^n$ is asymptotic to $f(x)$ as $x \to 0$".

Consider the integral

$$I(\lambda) \;=\; \int_0^A e^{-\lambda x} g(x) dx,$$

where $A > 0$, λ is real and g has the asymptotic expansion

$$g(x) \sim a_0 x^\alpha + a_1 x^{\alpha+1} + a_2 x^{\alpha+2} + \ldots$$

as $x \to +0$, with $\alpha > -1$. State Watson's lemma describing the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$, and determine an expression for the general term in the asymptotic series.

(ii) Let

$$h(t) = \pi^{-1/2} \int_0^\infty \frac{e^{-x}}{x^{1/2}(1+2xt)} dx$$

for $t \geqslant 0$. Show that

$$h(t) \;\sim\; \sum_{k=0}^{\infty} (-1)^k 1.3. \cdots .(2k-1) t^k$$

as $t \to +0$.

Suggest, for the case that t is smaller than unity, the point at which this asymptotic series should be truncated so as to produce optimal numerical accuracy.

18B Nonlinear Waves and Integrable Systems

(i) Write down a Lax pair for the equation

$$iq_t + q_{xx} = 0.$$

Discuss briefly, without giving mathematical details, how this pair can be used to solve the Cauchy problem on the infinite line for this equation. Discuss how this approach can be used to solve the analogous problem for the nonlinear Schrödinger equation.

(ii) Let $q(\zeta, \eta), \tilde{q}(\zeta, \eta)$ satisfy the equations

$$\tilde{q}_\zeta = q_\zeta + 2\lambda \sin \frac{\tilde{q} + q}{2}$$
$$\tilde{q}_\eta = -q_\eta + \frac{2}{\lambda} \sin \frac{\tilde{q} - q}{2},$$

where λ is a constant.

(a) Show that the above equations are compatible provided that $q, \tilde{q}$ both satisfy the Sine–Gordon equation

$$q_{\zeta\eta} = \sin q\,.$$

(b) Use the above result together with the fact that

$$\int \frac{dx}{\sin x} = \ln\left(\tan \frac{x}{2}\right) + \text{constant}\ ,$$

to show that the one-soliton solution of the Sine–Gordon equation is given by

$$\tan \frac{q}{4} = c \exp\left(\lambda\zeta + \frac{\eta}{\lambda}\right),$$

where c is a constant.

19E Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right), \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

with the initial condition $u(x,0) = \phi(x), 0 \leqslant x \leqslant 1$ and zero boundary conditions at $x = 0$ and $x = 1$, is solved by the finite-difference method

$$u_m^{n+1} = u_m^n + \mu[a_{m-\frac{1}{2}}u_{m-1}^n - (a_{m-\frac{1}{2}} + a_{m+\frac{1}{2}})u_m^n + a_{m+\frac{1}{2}}u_{m+1}^n], \\ m = 1, 2, \ldots, N,$$

where $\mu = \Delta t/(\Delta x)^2$, $\quad \Delta x = \frac{1}{N+1}$ and $u_m^n \approx u(m\Delta x, n\Delta t)$, $a_\alpha = a(\alpha\Delta x)$.

Assuming sufficient smoothness of the function a, and that μ remains constant as $\Delta x > 0$ and $\Delta t > 0$ become small, prove that the exact solution satisfies the numerical scheme with error $O((\Delta x)^3)$.

(ii) For the problem defined in Part (i), assume that there exist $0 < a_- < a_+ < \infty$ such that $a_- \leqslant a(x) \leqslant a_+$, $\quad 0 \leqslant x \leqslant 1$. Prove that the method is stable for $0 < \mu \leqslant 1/(2a_+)$.

[*Hint: You may use without proof the Gerschgorin theorem: All the eigenvalues of the matrix* $A = (a_{k,l})_{k,l=1,\ldots,M}$ *are contained in* $\bigcup_{k=1}^m \mathbb{S}_k$*, where*

$$\mathbb{S}_k = \left\{z \in \mathbb{C} : |z - a_{k,k}| \leqslant \sum_{\substack{l=1 \\ l \neq k}}^{m} |a_{k,l}|\right\}, \quad k = 1, 2, \ldots, m\,. \quad]$$