UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 4 June 2003 9 to 12

PAPER 2

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)***, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)***. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **16E, 19E** *should be in one bundle and* **4F, 8F** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) What is meant by a Poisson process of rate λ? Show that if $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ are independent Poisson processes of rates λ and μ respectively, then $(X_t + Y_t)_{t\geqslant 0}$ is also a Poisson process, and determine its rate.

(ii) A Poisson process of rate λ is observed by someone who believes that the first holding time is longer than all subsequent holding times. How long on average will it take before the observer is proved wrong?

2D Principles of Dynamics

(i) The trajectory $\mathbf{x}(t)$ of a non-relativistic particle of mass m and charge q moving in an electromagnetic field obeys the Lorentz equation

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \frac{\dot{\mathbf{x}}}{c} \wedge \mathbf{B}) .$$

Show that this equation follows from the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\left(\phi - \frac{\dot{\mathbf{x}}\cdot\mathbf{A}}{c}\right)$$

where $\phi(\mathbf{x}, t)$ is the electromagnetic scalar potential and $\mathbf{A}(\mathbf{x}, t)$ the vector potential, so that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi \text{ and } \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} .$$

(ii) Let $\mathbf{E} = 0$. Consider a particle moving in a constant magnetic field which points in the z direction. Show that the particle moves in a helix about an axis pointing in the z direction. Evaluate the radius of the helix.

3G Functional Analysis

(i) Define the dual of a normed vector space $(E, ||\cdot||)$. Show that the dual is always a complete normed space.

Prove that the vector space ℓ_1, consisting of those real sequences $(x_n)_{n=1}^{\infty}$ for which the norm

$$||(x_n)||_1 = \sum_{n=1}^{\infty} |x_n|$$

is finite, has the vector space ℓ_∞ of all bounded sequences as its dual.

(ii) State the Stone–Weierstrass approximation theorem.

Let K be a compact subset of $\mathbb{R}^n$. Show that every $f \in C_{\mathbb{R}}(K)$ can be uniformly approximated by a sequence of polynomials in n variables.

Let f be a continuous function on $[0,1] \times [0,1]$. Deduce that

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx\,.$$

4F Groups, Rings and Fields

(i) In each of the following two cases, determine a highest common factor in $\mathbb{Z}[i]$:

(a) $3+4i,\ 4-3i$;

(b) $3+4i,\ 1+2i$.

(ii) State and prove the Eisenstein criterion for irreducibility of polynomials with integer coefficients. Show that, if p is prime, the polynomial

$$1 + x + \cdots + x^{p-1}$$

is irreducible over $\mathbb{Z}$.

5A Electromagnetism

(i) A plane electromagnetic wave has electric and magnetic fields

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \qquad \mathbf{B} = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad (*)$$

for constant vectors $\mathbf{E}_0, \mathbf{B}_0$, constant positive angular frequency ω and constant wave-vector $\mathbf{k}$. Write down the vacuum Maxwell equations and show that they imply

$$\mathbf{k}\cdot\mathbf{E}_0 = 0\,, \qquad \mathbf{k}\cdot\mathbf{B}_0 = 0\,, \qquad \omega\mathbf{B}_0 = \mathbf{k}\times\mathbf{E}_0\,.$$

Show also that $|\mathbf{k}| = \omega/c$, where c is the speed of light.

(ii) State the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at the surface S of a perfect conductor. Let σ be the surface charge density and $\mathbf{s}$ the surface current density on S. How are σ and $\mathbf{s}$ related to $\mathbf{E}$ and $\mathbf{B}$?

A plane electromagnetic wave is incident from the half-space $x < 0$ upon the surface $x = 0$ of a perfectly conducting medium in $x > 0$. Given that the electric and magnetic fields of the incident wave take the form $(*)$ with

$$\mathbf{k} = k(\cos\theta, \sin\theta, 0) \qquad (0 < \theta < \pi/2)$$

and

$$\mathbf{E}_0 = \lambda\,(-\sin\theta, \cos\theta, 0)\,,$$

find $\mathbf{B}_0$.

Reflection of the incident wave at $x = 0$ produces a reflected wave with electric field

$$\mathbf{E}_0'\, e^{i\left(\mathbf{k}'\cdot\mathbf{r}-\omega t\right)}$$

with

$$\mathbf{k}' = k(-\cos\theta, \sin\theta, 0).$$

By considering the boundary conditions at $x = 0$ on the total electric field, show that

$$\mathbf{E}_0' = -\lambda\,(\sin\theta, \cos\theta, 0)\,.$$

Show further that the electric charge density on the surface $x = 0$ takes the form

$$\sigma = \sigma_0 e^{ik(y\sin\theta - ct)}$$

for a constant σ_0 that you should determine. Find the magnetic field of the reflected wave and hence the surface current density $\mathbf{s}$ on the surface $x = 0$.

6D Dynamics of Differential Equations

(i) What is a *Liapunov function*?

Consider the second order ODE

$$\dot{x} = y \ , \quad \dot{y} = -y - \sin^3 x \, .$$

By finding a suitable Liapunov function of the form $V(x, y) = f(x) + g(y)$, where f and g are to be determined, show that the origin is asymptotically stable. Using your form of V, find the greatest value of y_0 such that a trajectory through $(0, y_0)$ is guaranteed to tend to the origin as $t \to \infty$.

[*Any theorems you use need not be proved but should be clearly stated.*]

(ii) Explain the use of the stroboscopic method for investigating the dynamics of equations of the form $\ddot{x} + x = \epsilon f(x, \dot{x}, t)$, when $|\epsilon| \ll 1$. In particular, for $x = R\cos(t+\theta)$, $\dot{x} = -R\sin(t+\theta)$ derive the equations, correct to order ϵ,

$$\dot{R} = -\epsilon\langle f \sin(t+\theta)\rangle \ , \quad R\dot{\theta} = -\epsilon\langle f\cos(t+\theta)\rangle, \qquad (*)$$

where the brackets denote an average over the period of the unperturbed oscillator.

Find the form of the right hand sides of these equations explicitly when $f = \Gamma x^2 \cos t - 3qx$, where $\Gamma > 0$, $q \neq 0$. Show that apart from the origin there is another fixed point of $(*)$, and determine its stability. Sketch the trajectories in (R, θ) space in the case $q > 0$. What do you deduce about the dynamics of the full equation?

[*You may assume that* $\langle\cos^2 t\rangle = \frac{1}{2}$, $\langle\cos^4 t\rangle = \frac{3}{8}$, $\langle\cos^2 t \sin^2 t\rangle = \frac{1}{8}$.]

7H Geometry of Surfaces

(i) What are geodesic polar coordinates at a point P on a surface M with a Riemannian metric ds^2 ?

Assume that

$$ds^2 = dr^2 + H(r,\theta)^2 d\theta^2,$$

for geodesic polar coordinates r, θ and some function H. What can you say about H and dH/dr at $r = 0$?

(ii) Given that the Gaussian curvature K may be computed by the formula $K = -H^{-1}\partial^2 H/\partial r^2$, show that for small R the area of the geodesic disc of radius R centred at P is

$$\pi R^2 - (\pi/12)KR^4 + a(R),$$

where $a(R)$ is a function satisfying $\lim_{R\to 0} a(R)/R^4 = 0$.

8F Graph Theory

(i) State and prove a result of Euler relating the number of vertices, edges and faces of a plane graph. Use this result to exhibit a non-planar graph.

(ii) State the vertex form of Menger's Theorem and explain how it follows from an appropriate version of the Max-flow-min-cut Theorem. Let $k \geqslant 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

9F Coding and Cryptography

(i) Answer the following questions briefly but clearly.

(a) How does coding theory apply when the error rate $p > 1/2$?

(b) Give an example of a code which is not a linear code.

(c) Give an example of a linear code which is not a cyclic code.

(d) Give an example of a general feedback register with output k_j, and initial fill $(k_0, k_1, \ldots, k_N)$, such that

$$(k_n, k_{n+1}, \ldots, k_{n+N}) \neq (k_0, k_1, \ldots, k_N)$$

for all $n \geqslant 1$.

(e) Explain why the original Hamming code can not always correct two errors.

(ii) Describe the Rabin–Williams scheme for coding a message x as x^2 modulo a certain N. Show that, if N is chosen appropriately, breaking this code is equivalent to factorising the product of two primes.

10I Algorithms and Networks

(i) Consider a network with node set N and set of directed arcs A equipped with functions $d^+ : A \to \mathbb{Z}$ and $d^- : A \to \mathbb{Z}$ with $d^- \leqslant d^+$. Given $u : N \to \mathbb{R}$ we define the differential $\Delta u : A \to \mathbb{R}$ by $\Delta u(j) = u(i') - u(i)$ for $j = (i, i') \in A$. We say that Δu is a feasible differential if

$$d^-(j) \leqslant \Delta u(j) \leqslant d^+(j) \text{ for all } j \in A.$$

Write down a necessary and sufficient condition on d^+, d^- for the existence of a feasible differential and prove its necessity.

Assuming Minty's Lemma, describe an algorithm to construct a feasible differential and outline how this algorithm establishes the sufficiency of the condition you have given.

(ii) Let $E \subseteq S \times T$, where S, T are finite sets. A *matching* in E is a subset $M \subseteq E$ such that, for all $s, s' \in S$ and $t, t' \in T$,

$$\begin{aligned} (s,t), (s',t) \in M \quad & \text{implies} \quad s = s' \\ (s,t), (s,t') \in M \quad & \text{implies} \quad t = t'\,. \end{aligned}$$

A matching M is maximal if for any other matching M' with $M \subseteq M'$ we must have $M = M'$. Formulate the problem of finding a maximal matching in E in terms of an optimal distribution problem on a suitably defined network, and hence in terms of a standard linear optimization problem.

[*You may assume that the optimal distribution subject to integer constraints is integer-valued.*]

11I Principles of Statistics

(i) Outline briefly the Bayesian approach to hypothesis testing based on Bayes factors.

(ii) Let Y_1, Y_2 be independent random variables, both uniformly distributed on $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Find a minimal sufficient statistic for θ. Let $Y_{(1)} = \min\{Y_1, Y_2\}$, $Y_{(2)} = \max\{Y_1, Y_2\}$. Show that $R = Y_{(2)} - Y_{(1)}$ is ancilliary and explain why the Conditionality Principle would lead to inference about θ being drawn from the conditional distribution of $\frac{1}{2}\{Y_{(1)} + Y_{(2)}\}$ given R. Find the form of this conditional distribution.

12I Computational Statistics and Statistical Modelling

(i) Suppose $Y_1, \ldots, Y_n$ are independent Poisson variables, and

$$\mathbb{E}(Y_i) = \mu_i\,, \qquad \log \mu_i = \alpha + \beta t_i\,, \qquad \text{for} \quad i = 1, \ldots, n\,,$$

where α, β are two unknown parameters, and $t_1, \ldots, t_n$ are given covariates, each of dimension 1. Find equations for $\hat{\alpha}, \hat{\beta}$, the maximum likelihood estimators of α, β, and show how an estimate of $\text{var}(\hat{\beta})$ may be derived, quoting any standard theorems you may need.

(ii) By 31 December 2001, the number of new vCJD patients, classified by reported calendar year of onset, were

$$8, 10, 11, 14, 17, 29, 23$$

for the years

$$1994, \ldots, 2000 \quad \text{respectively.}$$

Discuss carefully the (slightly edited) R output for these data given below, quoting any standard theorems you may need.

```
> year
year
[1] 1994 1995 1996 1997 1998 1999 2000
> tot
[1]  8 10 11 14 17 29 23
>first.glm _ glm(tot ~ year, family = poisson)
> summary(first.glm)
Call:
glm(formula = tot ~ year, family = poisson)
Coefficients
                    Estimate Std. Error z value Pr(>|z|)
      (Intercept) -407.81285   99.35366  -4.105 4.05e-05
      year           0.20556    0.04973   4.133 3.57e-05


      (Dispersion parameter for poisson family taken to be 1)


      Null deviance: 20.7753  on 6  degrees of freedom
  Residual deviance:  2.7931  on 5  degrees of freedom


  Number of Fisher Scoring iterations: 3
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13C Foundations of Quantum Mechanics

(i) Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture. Explain how the two pictures provide equivalent descriptions of observable results.

Derive the equation of motion for an operator in the Heisenberg picture.

(ii) For a particle moving in one dimension, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, and the state vector is $|\Psi\rangle$. Eigenstates of $\hat{x}$ and $\hat{p}$ satisfy

$$\langle x|p\rangle = \left(\frac{1}{2\pi\hbar}\right)^{1/2} e^{ipx/\hbar}, \quad \langle x|x'\rangle = \delta(x-x'), \quad \langle p|p'\rangle = \delta(p-p')\,.$$

Use standard methods in the Dirac formalism to show that

$$\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x')$$
$$\langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p-p')\,.$$

Calculate $\langle x|\hat{H}|x'\rangle$ and express $\langle x|\hat{p}|\Psi\rangle$, $\langle x|\hat{H}|\Psi\rangle$ in terms of the position space wave function $\Psi(x)$.

Compute the momentum space Hamiltonian for the harmonic oscillator with potential $V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$.

14C Quantum Physics

(i) A system of N *distinguishable* non-interacting particles has energy levels E_i with degeneracy g_i, $1 \leqslant i < \infty$, for each particle. Show that in thermal equilibrium the number of particles N_i with energy E_i is given by

$$N_i = g_i e^{-\beta(E_i - \mu)},$$

where β and μ are parameters whose physical significance should be briefly explained.

A gas comprises a set of atoms with non-degenerate energy levels E_i, $1 \leqslant i < \infty$. Assume that the gas is dilute and the motion of the atoms can be neglected. For such a gas the atoms can be treated as distinguishable. Show that when the system is at temperature T, the number of atoms N_i at level i and the number N_j at level j satisfy

$$\frac{N_i}{N_j} = e^{-(E_i - E_j)/kT} \quad ,$$

where k is Boltzmann's constant.

(ii) A system of bosons has a set of energy levels W_a with degeneracy f_a, $1 \leqslant a < \infty$, for each particle. In thermal equilibrium at temperature T the number n_a of particles in level a is

$$n_a = \frac{f_a}{e^{(W_a - \mu)/kT} - 1} \quad .$$

What is the value of μ when the particles are photons?

Given that the density of states $\rho(\omega)$ for photons of frequency ω in a cubical box of side L is

$$\rho(\omega) = L^3 \frac{\omega^2}{\pi^2 c^3} \quad ,$$

where c is the speed of light, show that at temperature T the density of photons in the frequency range $\omega \to \omega + d\omega$ is $n(\omega) d\omega$ where

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1} \quad .$$

Deduce the energy density, $\epsilon(\omega)$, for photons of frequency ω .

The cubical box is occupied by the gas of atoms described in Part (i) in the presence of photons at temperature T . Consider the two atomic levels i and j where $E_i > E_j$ and $E_i - E_j = \hbar\omega$. The rate of spontaneous photon emission for the transition $i \to j$ is A_{ij}. The rate of absorption is $B_{ji}\, \epsilon(\omega)$ and the rate of stimulated emission is $B_{ij}\, \epsilon(\omega)$. Show that the requirement that these processes maintain the atoms and photons in thermal equilibrium yields the relations

$$B_{ij} = B_{ji}$$

and

$$A_{ij} = \left(\frac{\hbar\omega^3}{\pi^2 c^3} \right) B_{ij} \quad .$$

15A General Relativity

(i) What is a "stationary" metric? What distinguishes a stationary metric from a "static" metric?

A Killing vector field K^a of a metric g_{ab} satisfies

$$K_{a;b} + K_{b;a} = 0.$$

Show that this is equivalent to

$$g_{ab,c}K^c + g_{ac}K^c{}_{,b} + g_{cb}K^c{}_{,a} = 0.$$

Hence show that a constant vector field K^a with one non-zero component, K^4 say, is a Killing vector field if g_{ab} is independent of x^4.

(ii) Given that K^a is a Killing vector field, show that $K_a u^a$ is constant along the geodesic worldline of a massive particle with 4-velocity u^a. Hence find the energy ε of a particle of unit mass moving in a static spacetime with metric

$$ds^2 = h_{ij}dx^i dx^j - e^{2U}dt^2,$$

where h_{ij} and U are functions only of the space coordinates x^i. By considering a particle with speed small compared with that of light, and given that $U \ll 1$, show that $h_{ij} = \delta_{ij}$ to lowest order in the Newtonian approximation, and that U is the Newtonian potential.

A metric admits an antisymmetric tensor Y_{ab} satisfying

$$Y_{ab;c} + Y_{ac;b} = 0.$$

Given a geodesic $x^a(\lambda)$, let $s_a = Y_{ab}\,\dot{x}^b$. Show that s_a is parallelly propagated along the geodesic, and that it is orthogonal to the tangent vector of the geodesic. Hence show that the scalar

$$\phi = s^a s_a$$

is constant along the geodesic.

16E Theoretical Geophysics

(i) Explain briefly what is meant by the concepts of hydrostatic equilibrium and the buoyancy frequency. Evaluate an expression for the buoyancy frequency in an incompressible inviscid fluid with stable density profile $\rho(z)$.

(ii) Explain briefly what is meant by the Boussinesq approximation.

Write down the equations describing motions of small amplitude in an incompressible, stratified, Boussinesq fluid of constant buoyancy frequency.

Derive the resulting dispersion relationship for plane wave motion. Show that there is a maximum frequency for the waves and explain briefly why this is the case.

What would be the response to a solid body oscillating at a frequency in excess of the maximum?

17B Mathematical Methods

(i) Explain how to solve the Fredholm integral equation of the second kind,

$$f(x) = \mu \int_a^b K(x,t) f(t) dt \, + g(x) \, ,$$

in the case where $K(x,t)$ is of the separable (degenerate) form

$$K(x,t) = a_1(x) b_1(t) + a_2(x) b_2(t) \, .$$

(ii) For what values of the real constants λ and A does the equation

$$u(x) = \lambda \sin x \, + \, A \int_0^\pi (\cos x \cos t + \cos 2x \cos 2t)\, u(t) dt$$

have (a) a unique solution, (b) no solution?

18B Nonlinear Waves and Integrable Systems

(i) Write down the shock condition associated with the equation

$$\rho_t + q_x = 0,$$

where $q = q(\rho)$. Discuss briefly two possible heuristic approaches to justifying this shock condition.

(ii) According to shallow water theory, waves on a uniformly sloping beach are described by the equations

$$\begin{aligned}&\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0,\\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \quad h = \alpha x + \eta,\end{aligned}$$

where α is the constant slope of the beach, g is the gravitational acceleration, $u(x,t)$ is the fluid velocity, and $\eta(x,t)$ is the elevation of the fluid surface above the undisturbed level.

Find the characteristic velocities and the characteristic form of the equations.

What are the Riemann variables and how do they vary with t on the characteristics?

19E Numerical Analysis

(i) Explain briefly what is meant by the *convergence* of a numerical method for ordinary differential equations.

(ii) Suppose the sufficiently-smooth function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ obeys the Lipschitz condition: there exists $\lambda > 0$ such that

$$||\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})|| \leqslant \lambda ||\mathbf{x} - \mathbf{y}||, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, t \geqslant 0\,.$$

Prove from first principles, without using the Dahlquist equivalence theorem, that the trapezoidal rule

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]$$

for the solution of the ordinary differential equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \quad t \geqslant 0, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

converges.