UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Monday 3 June 2002 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***4D, 22D*** *should be in one bundle and* ***13L, 16L*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) We are given a finite set of airports. Assume that between any two airports, i and j, there are $a_{ij} = a_{ji}$ flights in each direction on every day. A confused traveller takes one flight per day, choosing at random from all available flights. Starting from i, how many days on average will pass until the traveller returns again to i? Be careful to allow for the case where there may be no flights at all between two given airports.

(ii) Consider the infinite tree T with root R, where, for all $m \geqslant 0$, all vertices at distance 2^m from R have degree 3, and where all other vertices (except R) have degree 2. Show that the random walk on T is recurrent.

2G Principles of Dynamics

(i) Derive Hamilton's equations from Lagrange's equations. Show that the Hamiltonian H is constant if the Lagrangian L does not depend explicitly on time.

(ii) A particle of mass m is constrained to move under gravity, which acts in the negative z-direction, on the spheroidal surface $\epsilon^{-2}(x^2 + y^2) + z^2 = l^2$, with $0 < \epsilon \leqslant 1$. If θ, ϕ parametrize the surface so that

$$x = \epsilon l \sin\theta \cos\phi, \;\; y = \epsilon l \sin\theta \sin\phi, \;\; z = l\cos\theta,$$

find the Hamiltonian $H(\theta, \phi, p_\theta, p_\phi)$.

Show that the energy

$$E = \frac{p_\theta^2}{2ml^2(\epsilon^2\cos^2\theta + \sin^2\theta)} + \frac{\alpha}{\sin^2\theta} + mgl\cos\theta$$

is a constant of the motion, where α is a non-negative constant.

Rewrite this equation as

$$\frac{1}{2}\dot{\theta}^2 + V_{\text{eff}}(\theta) = 0$$

and sketch $V_{\text{eff}}(\theta)$ for $\epsilon = 1$ and $\alpha > 0$, identifying the maximal and minimal values of $\theta(t)$ for fixed α and E. If ϵ is now taken not to be unity, how do these values depend on ϵ?

3H Groups, Rings and Fields

State Sylow's Theorems. Prove the existence part of Sylow's Theorems.

Show that any group of order 33 is cyclic.

Show that a group of order $p^2 q$, where p and q are distinct primes, is not simple. Is it always abelian? Give a proof or a counterexample.

4D Electromagnetism

(i) Show that, in a region where there is no magnetic field and the charge density vanishes, the electric field can be expressed either as minus the gradient of a scalar potential ϕ or as the curl of a vector potential $\mathbf{A}$. Verify that the electric field derived from

$$\mathbf{A} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\wedge\mathbf{r}}{r^3}$$

is that of an electrostatic dipole with dipole moment $\mathbf{p}$.

[*You may assume the following identities:*

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\wedge(\nabla\wedge\mathbf{b}) + \mathbf{b}\wedge(\nabla\wedge\mathbf{a}) + (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a},$$

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a}.]$$

(ii) An infinite conducting cylinder of radius a is held at zero potential in the presence of a line charge parallel to the axis of the cylinder at distance $s_0 > a$, with charge density q per unit length. Show that the electric field outside the cylinder is equivalent to that produced by replacing the cylinder with suitably chosen image charges.

5H Combinatorics

Prove that every graph G on $n \geqslant 3$ vertices with minimal degree $\delta(G) \geqslant \frac{n}{2}$ is Hamiltonian. For each $n \geqslant 3$, give an example to show that this result does not remain true if we weaken the condition to $\delta(G) \geqslant \frac{n}{2} - 1$ (n even) or $\delta(G) \geqslant \frac{n-1}{2}$ (n odd).

Now let G be a connected graph (with at least 2 vertices) without a cutvertex. Does G Hamiltonian imply G Eulerian? Does G Eulerian imply G Hamiltonian? Justify your answers.

6J Representation Theory

Construct the character table of the symmetric group S_5, explaining the steps in your construction.

Use the character table to show that the alternating group A_5 is the only non-trivial normal subgroup of S_5.

7J Galois Theory

Let $F \subset K$ be a finite extension of fields and let G be the group of F-automorphisms of K. State a result relating the order of G to the degree $[K : F]$.

Now let $K = k(X_1, \ldots, X_4)$ be the field of rational functions in four variables over a field k and let $F = k(s_1, \ldots, s_4)$ where $s_1, \ldots, s_4$ are the elementary symmetric polynomials in $k[X_1, \ldots, X_4]$. Show that the degree $[K : F] \leqslant 4!$ and deduce that F is the fixed field of the natural action of the symmetric group S_4 on K.

Show that $X_1X_3 + X_2X_4$ has a cubic minimum polynomial over F. Let $G = \langle \sigma, \tau \rangle \subset S_4$ be the dihedral group generated by the permutations $\sigma = (1234)$ and $\tau = (13)$. Show that the fixed field of G is $F(X_1X_3 + X_2X_4)$. Find the fixed field of the subgroup $H = \langle \sigma^2, \tau \rangle$.

8K Differentiable Manifolds

What is meant by a "bump function" on $\mathbb{R}^n$? If U is an open subset of a manifold M, prove that there is a bump function on M with support contained in U.

Prove the following.

(i) Given an open covering $\mathcal{U}$ of a compact manifold M, there is a partition of unity on M subordinate to $\mathcal{U}$.

(ii) Every compact manifold may be embedded in some Euclidean space.

9J Number Fields

Explain what is meant by an integral basis $\omega_1, \ldots, \omega_n$ of a number field K. Give an expression for the discriminant of K in terms of the traces of the $\omega_i\omega_j$.

Let $K = \mathbb{Q}(i, \sqrt{2})$. By computing the traces $T_{K/k}(\theta)$, where k runs through the three quadratic subfields of K, show that the algebraic integers θ in K have the form $\frac{1}{2}(\alpha + \beta\sqrt{2})$, where $\alpha = a + ib$ and $\beta = c + id$ are Gaussian integers. By further computing the norm $N_{K/k}(\theta)$, where $k = \mathbb{Q}(\sqrt{2})$, show that a and b are even and that $c \equiv d \pmod 2$. Hence prove that an integral basis for K is $1, i, \sqrt{2}, \frac{1}{2}(1 + i)\sqrt{2}$.

Calculate the discriminant of K.

10K Hilbert Spaces

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Define what it means for T to be *bounded below*. Prove that, if $LT = I$ for some $L \in \mathcal{B}(H)$, then T is bounded below.

Prove that an operator $T \in \mathcal{B}(H)$ is invertible if and only if both T and T^* are bounded below.

Let H be the sequence space ℓ^2. Define the operators S, R on H by setting

$$S(\xi) = (0, \xi_1, \xi_2, \xi_3, \dots), \quad R(\xi) = (\xi_2, \xi_3, \xi_4, \dots),$$

for all $\xi = (\xi_1, \xi_2, \xi_3, \dots) \in \ell^2$. Check that $RS = I$ but $SR \neq I$. Let $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. For each $\lambda \in D$, explain why $I - \lambda R$ is invertible, and define

$$R(\lambda) = (I - \lambda R)^{-1} R.$$

Show that, for all $\lambda \in D$, we have $R(\lambda)(S - \lambda I) = I$, but $(S - \lambda I)R(\lambda) \neq I$. Deduce that, for all $\lambda \in D$, the operator $S - \lambda I$ is bounded below, but is not invertible. Deduce also that $\operatorname{Sp} S = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and for $n = 1, 2, \dots$, define the element x_n of ℓ^2 by

$$x_n = n^{-1/2}(\lambda^{-1}, \lambda^{-2}, \dots, \lambda^{-n}, 0, 0, \dots).$$

Prove that $\|x_n\| = 1$ but that $(S - \lambda I)x_n \to 0$ as $n \to \infty$. Deduce that, for $|\lambda| = 1$, $S - \lambda I$ is not bounded below.

11K Riemann Surfaces

(a) Define the notions of (abstract) Riemann surface, holomorphic map, and biholomorphic map between Riemann surfaces.

(b) Prove the following theorem on the local form of a holomorphic map.

For a holomorphic map $f : R \to S$ between Riemann surfaces, which is not constant near a point $r \in R$, there exist neighbourhoods U of r in R and V of $f(r)$ in S, together with biholomorphic identifications $\phi : U \to \Delta$, $\psi : V \to \Delta$, such that $(\psi \circ f)(x) = \phi(x)^n$, for all $x \in U$.

(c) Prove further that a non-constant holomorphic map between compact, connected Riemann surfaces is surjective.

(d) Deduce from (c) the fundamental theorem of algebra.

12J Logic, Computation and Set Theory

(i) State the Knaster-Tarski fixed point theorem. Use it to prove the Cantor-Bernstein Theorem; that is, if there exist injections $A \to B$ and $B \to A$ for two sets A and B then there exists a bijection $A \to B$.

(ii) Let A be an arbitrary set and suppose given a subset R of $PA \times A$. We define a subset $B \subseteq A$ to be R-*closed* just if whenever $(S, a) \in R$ and $S \subseteq B$ then $a \in B$. Show that the set of all R-closed subsets of A is a complete poset in the inclusion ordering.

Now assume that A is itself equipped with a partial ordering $\leqslant$.

(a) Suppose R satisfies the condition that if $b \geqslant a \in A$ then $(\{b\}, a) \in R$.
Show that if B is R-closed then $c \leqslant b \in B$ implies $c \in B$.

(b) Suppose that R satisfies the following condition. Whenever $(S, a) \in R$ and $b \leqslant a$ then there exists $T \subseteq A$ such that $(T, b) \in R$, and for every $t \in T$ we have (i) $(\{b\}, t) \in R$, and (ii) $t \leqslant s$ for some $s \in S$. Let B and C be R-closed subsets of A. Show that the set

$$[B \to C] = \{a \in A \mid \forall b \leqslant a\, (b \in B \Rightarrow b \in C)\}$$

is R-closed.

13L Probability and Measure

State and prove Dynkin's π-system lemma.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (A_n) be a sequence of independent events such that $\lim_{n\to\infty} \mathbb{P}(A_n) = p$. Let $\mathcal{G} = \sigma(A_1, A_2, \ldots)$. Prove that

$$\lim_{n\to\infty} \mathbb{P}(G \cap A_n) = p\mathbb{P}(G)$$

for all $G \in \mathcal{G}$.

14M Information Theory

(a) Define the entropy $h(X)$ and the mutual entropy $i(X,Y)$ of random variables X and Y. Prove the inequality

$$0 \leqslant i(X,Y) \leqslant \min\{h(X), h(Y)\}.$$

[*You may assume the Gibbs inequality.*]

(b) Let X be a random variable and let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a random vector.

(i) Prove or disprove by producing a counterexample the inequality

$$i(X, \mathbf{Y}) \leqslant \sum_{j=1}^{n} i(X, Y_j),$$

first under the assumption that $Y_1, \ldots, Y_n$ are independent random variables, and then under the assumption that $Y_1, \ldots, Y_n$ are conditionally independent given X.

(ii) Prove or disprove by producing a counterexample the inequality

$$i(X, \mathbf{Y}) \geqslant \sum_{j=1}^{n} i(X, Y_j),$$

first under the assumption that $Y_1, \ldots, Y_n$ are independent random variables, and then under the assumption that $Y_1, \ldots, Y_n$ are conditionally independent given X.

15M Principles of Statistics

(i) Explain in detail the *minimax* and *Bayes* principles of decision theory.

Show that if $d(X)$ is a Bayes decision rule for a prior density $\pi(\theta)$ and has constant risk function, then $d(X)$ is minimax.

(ii) Let $X_1, \ldots, X_p$ be independent random variables, with $X_i \sim N(\mu_i, 1)$, $i = 1, \ldots, p$.

Consider estimating $\mu = (\mu_1, \ldots, \mu_p)^T$ by $d = (d_1, \ldots, d_p)^T$, with loss function

$$L(\mu, d) = \sum_{i=1}^{p} (\mu_i - d_i)^2 \ .$$

What is the risk function of $X = (X_1, \ldots, X_p)^T$?

Consider the class of estimators of μ of the form

$$d^a(X) = \left(1 - \frac{a}{X^T X}\right) X \ ,$$

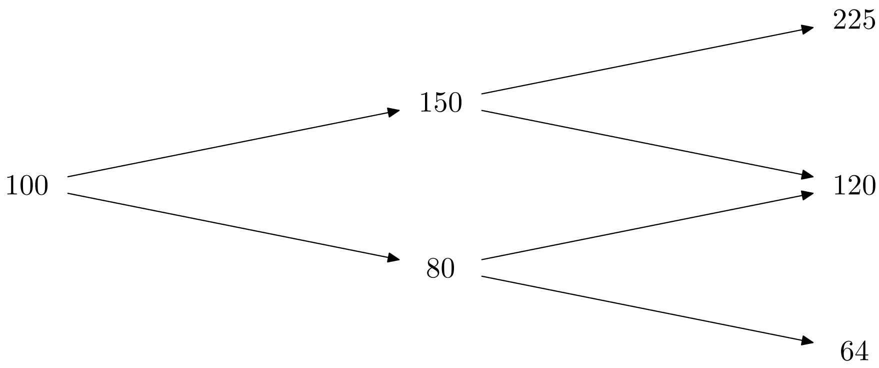
indexed by $a \geqslant 0$. Find the risk function of $d^a(X)$ in terms of $E\left(1/X^T X\right)$, which you should not attempt to evaluate, and deduce that X is inadmissible. What is the optimal value of a?

[*You may assume Stein's Lemma, that for suitably behaved real-valued functions h,*

$$E\left\{(X_i - \mu_i)h(X)\right\} = E\left\{\frac{\partial h(X)}{\partial X_i}\right\}. \quad]$$

16L Stochastic Financial Models

(i) The prices, S_i, of a stock in a binomial model at times $i = 0, 1, 2$ are represented by the following binomial tree.



The fixed interest rate per period is 1/5 and the probability that the stock price increases in a period is 1/3. Find the price at time 0 of a European call option with strike price 78 and expiry time 2.

Explain briefly the ideas underlying your calculations.

(ii) Consider an investor in a one-period model who may invest in s assets, all of which are risky, with a random return vector $\boldsymbol{R}$ having mean $\mathbb{E}\boldsymbol{R} = \boldsymbol{r}$ and positive-definite covariance matrix $\boldsymbol{V}$; assume that not all the assets have the same expected return. Show that any minimum-variance portfolio is equivalent to the investor dividing his wealth between two portfolios, the global minimum-variance portfolio and the diversified portfolio, both of which should be specified clearly in terms of $\boldsymbol{r}$ and $\boldsymbol{V}$.

Now suppose that $\boldsymbol{R} = (R_1, R_2, \ldots, R_s)^{\top}$ where $R_1, R_2, \ldots, R_s$ are independent random variables with R_i having the exponential distribution with probability density function $\lambda_i e^{-\lambda_i x}$, $x \geqslant 0$, where $\lambda_i > 0$, $1 \leqslant i \leqslant s$. Determine the global minimum-variance portfolio and the diversified portfolio explicitly.

Consider further the situation when the investor has the utility function $u(x) = 1 - e^{-x}$, where x denotes his wealth. Suppose that he acts to maximize the expected utility of his final wealth, and that his initial wealth is $w > 0$. Show that he now divides his wealth between the diversified portfolio and the *uniform* portfolio, in which wealth is apportioned equally between the assets, and determine the amounts that he invests in each.

17F Dynamical Systems

Let f_c be the map of the closed interval [0,1] to itself given by

$$f_c(x) = cx(1-x), \text{ where } 0 \leqslant c \leqslant 4.$$

Sketch the graphs of f_c and (without proof) of f_c^2, find their fixed points, and determine which of the fixed points of f_c are attractors. Does your argument work for $c = 3$?

18G Partial Differential Equations

(a) Solve the equation, for a function $u(x,y)$,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \qquad (*)$$

together with the boundary condition on the x-axis:

$$u(x,0)=x.$$

Find for which real numbers a it is possible to solve $(*)$ with the following boundary condition specified on the line $y=ax$:

$$u(x,ax)=x.$$

Explain your answer in terms of the notion of *characteristic hypersurface*, which should be defined.

(b) Solve the equation

$$\frac{\partial u}{\partial x}+(1+u)\frac{\partial u}{\partial y}=0$$

with the boundary condition on the x-axis

$$u(x,0)=x,$$

in the domain $\mathcal{D}=\{(x,y):0<y<(x+1)^2/4,\ \ -1<x<\infty\}$. Sketch the characteristics.

19G Methods of Mathematical Physics

State the Riemann-Lebesgue lemma as applied to the integral

$$\int_a^b g(u)e^{ixu}du \ ,$$

where $g'(u)$ is continuous and $a, b \in \mathbb{R}$.

Use this lemma to show that, as $x \to +\infty$,

$$\int_a^b (u-a)^{\lambda-1} f(u)e^{ixu}du \sim f(a)\, e^{ixa}\, e^{i\pi\lambda/2}\, \Gamma(\lambda)\, x^{-\lambda} \ ,$$

where $f(u)$ is holomorphic, $f(a) \neq 0$ and $0 < \lambda < 1$. You should explain each step of your argument, but detailed analysis is not required.

Hence find the leading order asymptotic behaviour as $x \to +\infty$ of

$$\int_0^1 \frac{e^{ixt^2}}{(1-t^2)^{\frac{1}{2}}}\, dt \ .$$

20F Numerical Analysis

(i) Let A be an $n \times n$ symmetric real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots., \mathbf{v}_n$, where $\|\mathbf{v}_l\| = 1$. Given $\mathbf{x}^{(0)} \in \mathbb{R}^n$, $\|\mathbf{x}^{(0)}\| = 1$, the sequence $\mathbf{x}^{(k)}$ is generated in the following manner. We set

$$\mu = \mathbf{x}^{(k)\,T} A \mathbf{x}^{(k)},$$

$$\mathbf{y} = (A - \mu I)^{-1} \mathbf{x}^{(k)},$$

$$\mathbf{x}^{(k+1)} = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

Show that if

$$\mathbf{x}^{(k)} = c^{-1} \left(\mathbf{v}_1 + \alpha \sum_{l=2}^{n} d_l \mathbf{v}_l \right),$$

where α is a real scalar and c is chosen so that $\|\mathbf{x}^{(k)}\| = 1$, then

$$\mu = c^{-2} \left(\lambda_1 + \alpha^2 \sum_{j=2}^{n} \lambda_j d_j^2 \right).$$

Give an explicit expression for c.

(ii) Use the above result to prove that, if $|\alpha|$ is small,

$$\mathbf{x}^{(k+1)} = \tilde{c}^{-1} \left(\mathbf{v}_1 + \alpha^3 \sum_{l=2}^{n} \tilde{d}_l \mathbf{v}_l \right) + O(\alpha^4)$$

and obtain the numbers $\tilde{c}$ and $\tilde{d}_2, \ldots, \tilde{d}_n$.

21E Electrodynamics

Explain how one can write Maxwell's equations in relativistic form by introducing an antisymmetric field strength tensor F_{ab}.

In an inertial frame S, the electric and magnetic fields are $\mathbf{E}$ and $\mathbf{B}$. Suppose that there is a second inertial frame S' moving with velocity v along the x-axis relative to S. Derive the rules for finding the electric and magnetic fields $\mathbf{E}'$ and $\mathbf{B}'$ in the frame S'. Show that $|\mathbf{E}|^2 - |\mathbf{B}|^2$ and $\mathbf{E} \cdot \mathbf{B}$ are invariant under Lorentz transformations.

Suppose that $\mathbf{E} = E_0(0\,, 1\,, 0)$ and $\mathbf{B} = E_0(0\,, \cos\theta\,, \sin\theta)$, where $0 \leq \theta < \pi/2$. At what velocity must an observer be moving in the frame S for the electric and magnetic fields to appear to be parallel?

Comment on the case $\theta = \pi/2$.

22D Statistical Physics

A simple model for a rubber molecule consists of a one-dimensional chain of n links each of fixed length b and each of which is oriented in either the positive or negative direction. A unique state i of the molecule is designated by giving the orientation ± 1 of each link. If there are n_+ links oriented in the positive direction and n_- links oriented in the negative direction then $n = n_+ + n_-$ and the length of the molecule is $l = (n_+ - n_-)b$. The length of the molecule associated with state i is l_i.

What is the range of l?

What is the number of states with n, n_+, n_- fixed?

Consider an ensemble of A copies of the molecule in which a_i members are in state i and write down the expression for the mean length L.

By introducing a Lagrange multiplier τ for L show that the most probable configuration for the $\{a_i\}$ with given length L is found by maximizing

$$\log\left(\frac{A!}{\prod_i a_i!}\right) + \tau \sum_i a_i l_i - \alpha \sum_i a_i .$$

Hence show that the most probable configuration is given by

$$p_i = \frac{e^{\tau l_i}}{Z} ,$$

where p_i is the probability for finding an ensemble member in the state i and Z is the partition function which should be defined.

Show that Z can be expressed as

$$Z = \sum_l g(l)\, e^{\tau l} ,$$

where the meaning of $g(l)$ should be explained.

Hence show that Z is given by

$$Z = \sum_{n_+=0}^{n} \frac{n!}{n_+!\, n_-!} (e^{\tau b})^{n_+} (e^{-\tau b})^{n_-} , \qquad n_+ + n_- = n,$$

and therefore that the free energy G for the system is

$$G = -nkT \log(2 \cosh \tau b) .$$

Show that τ is determined by

$$L = -\frac{1}{kT}\left(\frac{\partial G}{\partial \tau}\right)_n ,$$

and hence that the equation of state is

$$\tanh \tau b = \frac{L}{nb} .$$

What are the independent variables on which G depends?

Explain why the tension in the rubber molecule is $kT\tau$.

23E Applications of Quantum Mechanics

A quantum system, with Hamiltonian H_0, has continuous energy eigenstates $|E\rangle$ for all $E \geq 0$, and also a discrete eigenstate $|0\rangle$, with $H_0|0\rangle = E_0|0\rangle$, $\langle 0|0\rangle = 1$, $E_0 > 0$. A time-independent perturbation H_1, such that $\langle E|H_1|0\rangle \neq 0$, is added to H_0. If the system is initially in the state $|0\rangle$ obtain the formula for the decay rate

$$w = \frac{2\pi}{\hbar}\,\rho(E_0)\,\big|\langle E_0|H_1|0\rangle\big|^2\,,$$

where ρ is the density of states.

[*You may assume that* $\frac{1}{t}\Big(\frac{\sin\frac{1}{2}\omega t}{\frac{1}{2}\omega}\Big)^2$ *behaves like* $2\pi\,\delta(\omega)$ *for large* t.]

Assume that, for a particle moving in one dimension,

$$H_0 = E_0|0\rangle\langle 0| + \int_{-\infty}^{\infty} p^2 |p\rangle\langle p|\,dp\,, \qquad H_1 = f\int_{-\infty}^{\infty}\big(|p\rangle\langle 0| + |0\rangle\langle p|\big)\,dp\,,$$

where $\langle p'|p\rangle = \delta(p' - p)$, and f is constant. Obtain w in this case.

24D General Relativity

(i) Given a covariant vector field V_a, define the curvature tensor $R^a{}_{bcd}$ by

$$V_{a;bc} - V_{a;cb} = V_e R^e{}_{abc}. \qquad (*)$$

Express $R^e{}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$$R^e{}_{abc} = -R^e{}_{acb}.$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$$R^e{}_{abc} + R^e{}_{cab} + R^e{}_{bca} = 0.$$

(ii) Write down an expression similar to $(*)$ given in Part (i) for the quantity

$$g_{ab;cd} - g_{ab;dc}$$

and hence show that

$$R_{eabc} = -R_{aebc}.$$

Define the Ricci tensor, show that it is symmetric and write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geq 2$, R_{abcd} takes the form

$$R_{abcd} = K(x^e)[g_{ac}g_{bd} - g_{ad}g_{bc}].$$

Obtain the Ricci tensor and Ricci scalar. Deduce that K is a constant in such spacetimes if the dimension n is greater than 2.

25C Fluid Dynamics II

State the minimum dissipation theorem for Stokes flow in a bounded domain.

Fluid of density ρ and viscosity μ fills an infinite cylindrical annulus $a \leq r \leq b$ between a fixed cylinder $r = a$ and a cylinder $r = b$ which rotates about its axis with constant angular velocity Ω. In cylindrical polar coordinates (r, θ, z), the fluid velocity is $\mathbf{u} = (0, v(r), 0)$. The Reynolds number $\rho\Omega b^2/\mu$ is not necessarily small. Show that $v(r) = Ar + B/r$, where A and B are constants to be determined.

[*You may assume that* $\nabla^2\mathbf{u} = (0, \nabla^2 v - v/r^2, 0)$ *and* $(\mathbf{u}\cdot\nabla)\mathbf{u} = (-v^2/r,\, 0,\, 0)$.]

Show that the outer cylinder exerts a couple G_0 per unit length on the fluid, where

$$G_0 = \frac{4\pi\mu\Omega a^2 b^2}{b^2 - a^2}.$$

[*You may assume that, in standard notation,* $e_{r\theta} = \dfrac{r}{2}\dfrac{d}{dr}\left(\dfrac{v}{r}\right)$.]

Suppose now that $b \geq \sqrt{2}a$ and that the cylinder $r = a$ is replaced by a fixed cylinder whose cross-section is a square of side $2a$ centred on $r = 0$, all other conditions being unchanged. The flow may still be assumed steady. Explaining your argument carefully, show that the couple G now required to maintain the motion of the outer cylinder is greater than G_0.

26C Waves in Fluid and Solid Media

Starting from the equations governing sound waves linearized about a state with density ρ_0 and sound speed c_0, derive the acoustic energy equation, giving expressions for the local energy density E and energy flux $\mathbf{I}$.

A sphere executes small-amplitude vibrations, with its radius varying according to

$$r(t) = a + \mathrm{Re}\left(\epsilon e^{i\omega t}\right),$$

with $0 < \epsilon \ll a$. Find an expression for the velocity potential of the sound, $\tilde{\phi}(r, t)$. Show that the time-averaged rate of working by the surface of the sphere is

$$2\pi a^2 \rho_0 \omega^2 \epsilon^2 c_0 \frac{\omega^2 a^2}{c_0^2 + \omega^2 a^2}.$$

Calculate the value at $r = a$ of the dimensionless ratio $c_0\overline{E}/|\overline{\mathbf{I}}|$, where the overbars denote time-averaged values, and comment briefly on the limits $c_0 \ll \omega a$ and $c_0 \gg \omega a$.