UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 3 June 2002 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question, but must not attempt Parts from more than* ***SIX*** *questions. If you submit answers to Parts of more than six questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***C,D,E, . . . , M*** *according to the letter affixed to each question. (For example,* ***2G, 19G*** *should be in one bundle and* ***7J, 9J*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *Parts of* ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) We are given a finite set of airports. Assume that between any two airports, i and j, there are $a_{ij} = a_{ji}$ flights in each direction on every day. A confused traveller takes one flight per day, choosing at random from all available flights. Starting from i, how many days on average will pass until the traveller returns again to i? Be careful to allow for the case where there may be no flights at all between two given airports.

(ii) Consider the infinite tree T with root R, where, for all $m \geqslant 0$, all vertices at distance 2^m from R have degree 3, and where all other vertices (except R) have degree 2. Show that the random walk on T is recurrent.

2G Principles of Dynamics

(i) Derive Hamilton's equations from Lagrange's equations. Show that the Hamiltonian H is constant if the Lagrangian L does not depend explicitly on time.

(ii) A particle of mass m is constrained to move under gravity, which acts in the negative z-direction, on the spheroidal surface $\epsilon^{-2}(x^2 + y^2) + z^2 = l^2$, with $0 < \epsilon \leqslant 1$. If θ, ϕ parametrize the surface so that

$$x = \epsilon l \sin\theta \cos\phi, \;\; y = \epsilon l \sin\theta \sin\phi, \;\; z = l\cos\theta,$$

find the Hamiltonian $H(\theta, \phi, p_\theta, p_\phi)$.

Show that the energy

$$E = \frac{p_\theta^2}{2ml^2(\epsilon^2\cos^2\theta + \sin^2\theta)} + \frac{\alpha}{\sin^2\theta} + mgl\cos\theta$$

is a constant of the motion, where α is a non-negative constant.

Rewrite this equation as

$$\frac{1}{2}\dot{\theta}^2 + V_{\text{eff}}(\theta) = 0$$

and sketch $V_{\text{eff}}(\theta)$ for $\epsilon = 1$ and $\alpha > 0$, identifying the maximal and minimal values of $\theta(t)$ for fixed α and E. If ϵ is now taken not to be unity, how do these values depend on ϵ?

3K Functional Analysis

(i) Let $P_r(e^{i\theta})$ be the real part of $\dfrac{1+re^{i\theta}}{1-re^{i\theta}}$. Establish the following properties of P_r for $0 \leqslant r < 1$:

(a) $0 < P_r(e^{i\theta}) = P_r(e^{-i\theta}) \leqslant \dfrac{1+r}{1-r}$;

(b) $P_r(e^{i\theta}) \leqslant P_r(e^{i\delta})$ for $0 < \delta \leqslant |\theta| \leqslant \pi$;

(c) $P_r(e^{i\theta}) \to 0$, uniformly on $0 < \delta \leqslant |\theta| \leqslant \pi$, as r increases to 1.

(ii) Suppose that $f \in L^1(\mathbf{T})$, where $\mathbf{T}$ is the unit circle $\{e^{i\theta} : -\pi \leqslant \theta \leqslant \pi\}$. By definition, $\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta$. Let

$$P_r(f)(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i(\theta-t)}) f(e^{it})\, dt.$$

Show that $P_r(f)$ is a continuous function on $\mathbf{T}$, and that $\|P_r(f)\|_1 \leqslant \|f\|_1$.

[*You may assume without proof that* $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i\theta})\, d\theta = 1$.]

Show that $P_r(f) \to f$, uniformly on $\mathbf{T}$ as r increases to 1, if and only if f is a continuous function on $\mathbf{T}$.

Show that $\|P_r(f) - f\|_1 \to 0$ as r increases to 1.

4H Groups, Rings and Fields

(i) What is a Sylow subgroup? State Sylow's Theorems.

Show that any group of order 33 is cyclic.

(ii) Prove the existence part of Sylow's Theorems.

[*You may use without proof any arithmetic results about binomial coefficients which you need.*]

Show that a group of order p^2q, where p and q are distinct primes, is not simple. Is it always abelian? Give a proof or a counterexample.

5D Electromagnetism

(i) Show that, in a region where there is no magnetic field and the charge density vanishes, the electric field can be expressed either as minus the gradient of a scalar potential ϕ or as the curl of a vector potential $\mathbf{A}$. Verify that the electric field derived from

$$\mathbf{A} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\wedge\mathbf{r}}{r^3}$$

is that of an electrostatic dipole with dipole moment $\mathbf{p}$.

[*You may assume the following identities:*

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\wedge(\nabla\wedge\mathbf{b}) + \mathbf{b}\wedge(\nabla\wedge\mathbf{a}) + (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a},$$

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a}.]$$

(ii) An infinite conducting cylinder of radius a is held at zero potential in the presence of a line charge parallel to the axis of the cylinder at distance $s_0 > a$, with charge density q per unit length. Show that the electric field outside the cylinder is equivalent to that produced by replacing the cylinder with suitably chosen image charges.

6F Dynamics of Differential Equations

(i) A system in $\mathbb{R}^2$ obeys the equations:

$$\dot{x} = x - x^5 - 2xy^4 - 2y^3(a - x^2)\ ,$$
$$\dot{y} = y - x^4y - 2y^5 + x^3(a - x^2)\ ,$$

where a is a positive constant.

By considering the quantity $V = \alpha x^4 + \beta y^4$, where α and β are appropriately chosen, show that if $a > 1$ then there is a unique fixed point and a unique limit cycle. How many fixed points are there when $a < 1$?

(ii) Consider the second order system

$$\ddot{x} - (a - bx^2)\dot{x} + x - x^3 = 0\ ,$$

where a, b are constants.

(a) Find the fixed points and determine their stability.

(b) Show that if the fixed point at the origin is unstable and $3a > b$ then there are no limit cycles.

[*You may find it helpful to use the Liénard coordinate* $z = \dot{x} - ax + \frac{1}{3}bx^3$.]

7J Logic, Computation and Set Theory

(i) State the Knaster-Tarski fixed point theorem. Use it to prove the Cantor-Bernstein Theorem; that is, if there exist injections $A \to B$ and $B \to A$ for two sets A and B then there exists a bijection $A \to B$.

(ii) Let A be an arbitrary set and suppose given a subset R of $PA \times A$. We define a subset $B \subseteq A$ to be R-*closed* just if whenever $(S, a) \in R$ and $S \subseteq B$ then $a \in B$. Show that the set of all R-closed subsets of A is a complete poset in the inclusion ordering.

Now assume that A is itself equipped with a partial ordering $\leqslant$.

(a) Suppose R satisfies the condition that if $b \geqslant a \in A$ then $(\{b\}, a) \in R$. Show that if B is R-closed then $c \leqslant b \in B$ implies $c \in B$.

(b) Suppose that R satisfies the following condition. Whenever $(S, a) \in R$ and $b \leqslant a$ then there exists $T \subseteq A$ such that $(T, b) \in R$, and for every $t \in T$ we have (i) $(\{b\}, t) \in R$, and (ii) $t \leqslant s$ for some $s \in S$. Let B and C be R-closed subsets of A. Show that the set

$$[B \to C] = \{a \in A \mid \forall b \leqslant a\, (b \in B \Rightarrow b \in C)\}$$

is R-closed.

8H Graph Theory

(i) State and prove a necessary and sufficient condition for a graph to be Eulerian (that is, to have an Eulerian circuit).

Prove that, given any connected non-Eulerian graph G, there is an Eulerian graph H and a vertex $v \in H$ such that $G = H - v$.

(ii) Let G be a connected plane graph with n vertices, e edges and f faces. Prove that $n - e + f = 2$. Deduce that $e \le g(n-2)/(g-2)$, where g is the smallest face size.

The *crossing number* $c(G)$ of a non-planar graph G is the minimum number of edge-crossings needed when drawing the graph in the plane. (The crossing of three edges at the same point is not allowed.) Show that if G has n vertices and e edges then $c(G) \ge e - 3n + 6$. Find $c(K_6)$.

9J Number Theory

(i) Let p be a prime number. Prove that the multiplicative group of the field with p elements is cyclic.

(ii) Let p be an odd prime, and let $k \geqslant 1$ be an integer. Prove that we have $x^2 \equiv 1 \bmod p^k$ if and only if either $x \equiv 1 \bmod p^k$ or $x \equiv -1 \bmod p^k$. Is this statement true when $p = 2$?

Let m be an odd positive integer, and let r be the number of distinct prime factors of m. Prove that there are precisely 2^r different integers x satisfying $x^2 \equiv 1 \bmod m$ and $0 < x < m$.

10H Coding and Cryptography

(i) Describe the original Hamming code of length 7. Show how to encode a message word, and how to decode a received word involving at most one error. Explain why the procedure works.

(ii) What is a linear binary code? What is its dual code? What is a cyclic binary code? Explain how cyclic binary codes of length n correspond to polynomials in $\mathbb{F}_2[X]$ dividing $X^n + 1$. Show that the dual of a cyclic code of length n is cyclic of length n.

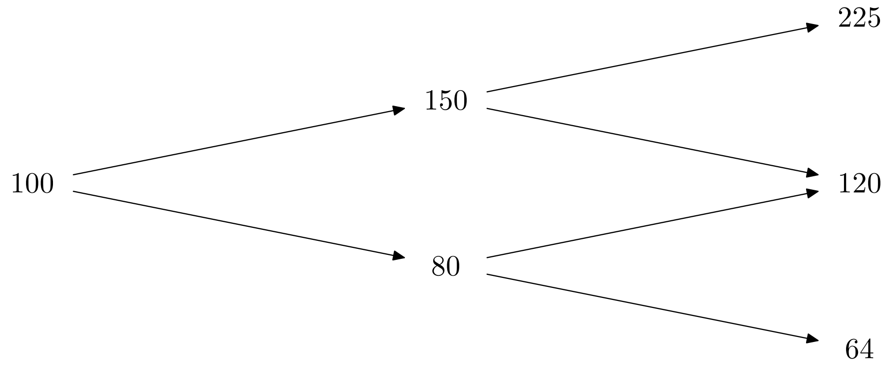
Using the factorization

$$X^7 + 1 = (X + 1)(X^3 + X + 1)(X^3 + X^2 + 1)$$

in $\mathbb{F}_2[X]$, find all cyclic binary codes of length 7. Identify those which are Hamming codes and their duals. Justify your answer.

11L Stochastic Financial Models

(i) The prices, S_i, of a stock in a binomial model at times $i = 0, 1, 2$ are represented by the following binomial tree.



The fixed interest rate per period is 1/5 and the probability that the stock price increases in a period is 1/3. Find the price at time 0 of a European call option with strike price 78 and expiry time 2.

Explain briefly the ideas underlying your calculations.

(ii) Consider an investor in a one-period model who may invest in s assets, all of which are risky, with a random return vector $\boldsymbol{R}$ having mean $\mathbb{E}\boldsymbol{R} = \boldsymbol{r}$ and positive-definite covariance matrix $\boldsymbol{V}$; assume that not all the assets have the same expected return. Show that any minimum-variance portfolio is equivalent to the investor dividing his wealth between two portfolios, the global minimum-variance portfolio and the diversified portfolio, both of which should be specified clearly in terms of $\boldsymbol{r}$ and $\boldsymbol{V}$.

Now suppose that $\boldsymbol{R} = (R_1, R_2, \ldots, R_s)^{\top}$ where $R_1, R_2, \ldots, R_s$ are independent random variables with R_i having the exponential distribution with probability density function $\lambda_i e^{-\lambda_i x}$, $x \geqslant 0$, where $\lambda_i > 0$, $1 \leqslant i \leqslant s$. Determine the global minimum-variance portfolio and the diversified portfolio explicitly.

Consider further the situation when the investor has the utility function $u(x) = 1 - e^{-x}$, where x denotes his wealth. Suppose that he acts to maximize the expected utility of his final wealth, and that his initial wealth is $w > 0$. Show that he now divides his wealth between the diversified portfolio and the *uniform* portfolio, in which wealth is apportioned equally between the assets, and determine the amounts that he invests in each.

12M Principles of Statistics

(i) Explain in detail the *minimax* and *Bayes* principles of decision theory.

Show that if $d(X)$ is a Bayes decision rule for a prior density $\pi(\theta)$ and has constant risk function, then $d(X)$ is minimax.

(ii) Let $X_1, \ldots, X_p$ be independent random variables, with $X_i \sim N(\mu_i, 1)$, $i = 1, \ldots, p$.

Consider estimating $\mu = (\mu_1, \ldots, \mu_p)^T$ by $d = (d_1, \ldots, d_p)^T$, with loss function

$$L(\mu, d) = \sum_{i=1}^{p} (\mu_i - d_i)^2 \ .$$

What is the risk function of $X = (X_1, \ldots, X_p)^T$?

Consider the class of estimators of μ of the form

$$d^a(X) = \left(1 - \frac{a}{X^T X}\right) X \ ,$$

indexed by $a \geqslant 0$. Find the risk function of $d^a(X)$ in terms of $E\left(1/X^T X\right)$, which you should not attempt to evaluate, and deduce that X is inadmissible. What is the optimal value of a?

[*You may assume Stein's Lemma, that for suitably behaved real-valued functions h,*

$$E\left\{(X_i - \mu_i) h(X)\right\} = E\left\{\frac{\partial h(X)}{\partial X_i}\right\} . \quad]$$

13L Computational Statistics and Statistical Modelling

(i) Suppose $Y_1, \ldots, Y_n$ are independent Poisson variables, and

$$\mathbb{E}(Y_i) = \mu_i \ , \ \log \mu_i = \alpha + \beta^T x_i \ , \ 1 \leqslant i \leqslant n$$

where α, β are unknown parameters, and $x_1, \ldots, x_n$ are given covariates, each of dimension p. Obtain the maximum-likelihood equations for α, β, and explain briefly how you would check the validity of this model.

(ii) The data below show $y_1, \ldots, y_{33}$, which are the monthly accident counts on a major US highway for each of the 12 months of 1970, then for each of the 12 months of 1971, and finally for the first 9 months of 1972. The data-set is followed by the (slightly edited) R output. You may assume that the factors 'Year' and 'month' have been set up in the appropriate fashion. Give a careful interpretation of this R output, and explain (a) how you would derive the corresponding standardised residuals, and (b) how you would predict the number of accidents in October 1972.

```
52 37  49  29  31  32  28  34  32  39  50  63
35 22  27  27  34  23  42  30  36  56  48  40
33 26  31  25  23  20  25  20  36
```

> first.glm _ glm(y~ Year + month, poisson) ; summary(first.glm)

Call:

glm(formula = y ~ Year + month, family = poisson)

Coefficients:

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	3.81969	0.09896	38.600	< 2e − 16 ***
Year1971	-0.12516	0.06694	-1.870	0.061521 .
Year1972	-0.28794	0.08267	-3.483	0.000496 ***
month2	-0.34484	0.14176	-2.433	0.014994 *
month3	-0.11466	0.13296	-0.862	0.388459
month4	-0.39304	0.14380	-2.733	0.006271 **
month5	-0.31015	0.14034	-2.210	0.027108 *
month6	-0.47000	0.14719	-3.193	0.001408 **
month7	-0.23361	0.13732	-1.701	0.088889 .
month8	-0.35667	0.14226	-2.507	0.012168 *
month9	-0.14310	0.13397	-1.068	0.285444
month10	0.10167	0.13903	0.731	0.464628
month11	0.13276	0.13788	0.963	0.335639
month12	0.18252	0.13607	1.341	0.179812

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

(Dispersion parameter for poisson family taken to be 1)

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    Null deviance:     101.143   on 32 degrees of freedom
Residual deviance:     27.273    on 19 degrees of freedom
```

Number of Fisher Scoring iterations: 3

14E Quantum Physics

(i) A system of N identical non-interacting bosons has energy levels E_i with degeneracy g_i, $1 \leq i < \infty$, for each particle. Show that in thermal equilibrium the number of particles N_i with energy E_i is given by

$$N_i = \frac{g_i}{e^{\beta(E_i - \mu)} - 1} \quad ,$$

where β and μ are parameters whose physical significance should be briefly explained.

(ii) A photon moves in a cubical box of side L. Assuming periodic boundary conditions, show that, for large L, the number of photon states lying in the frequency range $\omega \to \omega + d\omega$ is $\rho(\omega) d\omega$ where

$$\rho(\omega) = L^3 \left(\frac{\omega^2}{\pi^2 c^3} \right) \quad .$$

If the box is filled with thermal radiation at temperature T, show that the number of photons per unit volume in the frequency range $\omega \to \omega + d\omega$ is $n(\omega) d\omega$ where

$$n(\omega) = \left(\frac{\omega^2}{\pi^2 c^3} \right) \frac{1}{e^{\hbar\omega/kT} - 1} \quad .$$

Calculate the energy density W of the thermal radiation. Show that the pressure P exerted on the surface of the box satisfies

$$P = \frac{1}{3} W \quad .$$

[*You may use the result* $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$.]

15D General Relativity

(i) Given a covariant vector field V_a, define the curvature tensor $R^a{}_{bcd}$ by

$$V_{a;bc} - V_{a;cb} = V_e R^e{}_{abc}. \qquad (*)$$

Express $R^e{}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$$R^e{}_{abc} = -R^e{}_{acb}.$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$$R^e{}_{abc} + R^e{}_{cab} + R^e{}_{bca} = 0.$$

(ii) Write down an expression similar to $(*)$ given in Part (i) for the quantity

$$g_{ab;cd} - g_{ab;dc}$$

and hence show that

$$R_{eabc} = -R_{aebc}.$$

Define the Ricci tensor, show that it is symmetric and write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geq 2$, R_{abcd} takes the form

$$R_{abcd} = K(x^e)[g_{ac}g_{bd} - g_{ad}g_{bc}].$$

Obtain the Ricci tensor and Ricci scalar. Deduce that K is a constant in such spacetimes if the dimension n is greater than 2.

16D Statistical Physics and Cosmology

(i) Consider a one-dimensional model universe with "stars" distributed at random on the x-axis, and choose the origin to coincide with one of the stars; call this star the "home-star." Home-star astronomers have discovered that all other stars are receding from them with a velocity $v(x)$, that depends on the position x of the star. Assuming non-relativistic addition of velocities, show how the assumption of homogeneity implies that $v(x) = H_0 x$ for some constant H_0.

In attempting to understand the history of their one-dimensional universe, home-star astronomers seek to determine the velocity $v(t)$ at time t of a star at position $x(t)$. Assuming homogeneity, show how $x(t)$ is determined in terms of a scale factor $a(t)$ and hence deduce that $v(t) = H(t)x(t)$ for some function $H(t)$. What is the relation between $H(t)$ and H_0?

(ii) Consider a three-dimensional homogeneous and isotropic universe with mass density $\rho(t)$, pressure $p(t)$ and scale factor $a(t)$. Given that $E(t)$ is the energy in volume $V(t)$, show how the relation $dE = -p\,dV$ yields the "fluid" equation

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right) H,$$

where $H = \dot{a}/a$.

Show how conservation of energy applied to a test particle at the boundary of a spherical fluid element yields the Friedmann equation

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = -kc^2$$

for constant k. Hence obtain an equation for the acceleration $\ddot{a}$ in terms of ρ, p and a.

A model universe has mass density and pressure

$$\rho = \frac{\rho_0}{a^3} + \rho_1, \qquad p = -\rho_1 c^2,$$

where ρ_0 is constant. What does the fluid equation imply about ρ_1? Show that the acceleration $\ddot{a}$ vanishes if

$$a = \left(\frac{\rho_0}{2\rho_1}\right)^{\frac{1}{3}}.$$

Hence show that this universe is static and determine the sign of the constant k.

17E Symmetries and Groups in Physics

(i) Let H be a normal subgroup of the group G. Let G/H denote the group of cosets $\tilde{g} = gH$ for $g \in G$. If $D : G \to GL(\mathbb{C}^n)$ is a representation of G with $D(h_1) = D(h_2)$ for all $h_1, h_2 \in H$ show that $\tilde{D}(\tilde{g}) = D(g)$ is well-defined and that it is a representation of G/H. Show further that $\tilde{D}(\tilde{g})$ is irreducible if and only if $D(g)$ is irreducible.

(ii) For a matrix $U \in SU(2)$ define the linear map $\Phi_U : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_U(\mathbf{x}).\boldsymbol{\sigma} = U\mathbf{x}.\boldsymbol{\sigma}U^\dagger$ with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ as the vector of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Show that $\|\Phi_U(\mathbf{x})\| = \|\mathbf{x}\|$. Because of the linearity of Φ_U there exists a matrix $R(U)$ such that $\Phi_U(\mathbf{x}) = R(U)\mathbf{x}$. Given that any $SU(2)$ matrix can be written as

$$U = \cos\alpha\, I - i \sin\alpha\, \mathbf{n}.\boldsymbol{\sigma} \quad ,$$

where $\alpha \in [0, \pi]$ and $\mathbf{n}$ is a unit vector, deduce that $R(U) \in SO(3)$ for all $U \in SU(2)$. Compute $R(U)\mathbf{n}$ and $R(U)\mathbf{x}$ in the case that $\mathbf{x}.\mathbf{n} = 0$ and deduce that $R(U)$ is the matrix of a rotation about $\mathbf{n}$ with angle 2α.

[*Hint:* $\mathbf{m}.\boldsymbol{\sigma}\ \mathbf{n}.\boldsymbol{\sigma} = \mathbf{m}.\mathbf{n}\, I + i(\mathbf{m} \times \mathbf{n}).\boldsymbol{\sigma}$.]

Show that $R(U)$ defines a surjective homomorphism $\Theta : SU(2) \to SO(3)$ and find the kernel of Θ.

18C Transport Processes

(i) Material of thermal diffusivity D occupies the semi-infinite region $x > 0$ and is initially at uniform temperature T_0. For time $t > 0$ the temperature at $x = 0$ is held at a constant value $T_1 > T_0$. Given that the temperature $T(x,t)$ in $x > 0$ satisfies the diffusion equation $T_t = DT_{xx}$, write down the equation and the boundary and initial conditions satisfied by the dimensionless temperature $\theta = (T - T_0)\,/\,(T_1 - T_0)$.

Use dimensional analysis to show that the lengthscale of the region in which T is significantly different from T_0 is proportional to $(Dt)^{1/2}$. Hence show that this problem has a similarity solution

$$\theta = \operatorname{erfc}(\xi/2) \equiv \frac{2}{\sqrt{\pi}} \int_{\xi/2}^{\infty} e^{-u^2} du \ ,$$

where $\xi = x/(Dt)^{1/2}$.

What is the rate of heat input, $-DT_x$, across the plane $x = 0$?

(ii) Consider the same problem as in Part (i) except that the boundary condition at $x = 0$ is replaced by one of constant rate of heat input Q. Show that $\theta(\xi, t)$ satisfies the partial differential equation

$$\theta_{\xi\xi} + \frac{\xi}{2}\theta_\xi = t\theta_t$$

and write down the boundary conditions on $\theta(\xi, t)$. Deduce that the problem has a similarity solution of the form

$$\theta = \frac{Q(t/D)^{1/2}}{T_1 - T_0} f(\xi).$$

Derive the ordinary differential equation and boundary conditions satisfied by $f(\xi)$. Differentiate this equation once to obtain

$$f''' + \frac{\xi}{2} f'' = 0$$

and solve for $f'(\xi)$. Hence show that

$$f(\xi) = \frac{2}{\sqrt{\pi}} e^{-\xi^2/4} - \xi \operatorname{erfc}(\xi/2) \ .$$

Sketch the temperature distribution $T(x,t)$ for various times t, and calculate $T(0,t)$ explicitly.

19G Theoretical Geophysics

(i) In a reference frame rotating about a vertical axis with constant angular velocity $f/2$ the horizontal components of the momentum equation for a shallow layer of inviscid, incompressible fluid of constant density ρ are

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad ,$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad ,$$

where u, v and P are independent of the vertical coordinate z.

Define the Rossby number Ro for a flow with typical velocity U and lengthscale L. What is the approximate form of the above equations when $Ro \ll 1$?

Show that the solution to the approximate equations is given by a streamfunction ψ proportional to P.

Conservation of potential vorticity for such a flow is represented by

$$\frac{D}{Dt}\frac{\zeta + f}{h} = 0,$$

where ζ is the vertical component of relative vorticity and $h(x, y)$ is the thickness of the layer. Explain briefly why the potential vorticity of a column of fluid should be conserved.

(ii) Suppose that the thickness of the rotating, shallow-layer flow in Part (i) is $h(y) = H_0 \exp(-\alpha y)$ where H_0 and α are constants. By linearising the equation of conservation of potential vorticity about $u = v = \zeta = 0$, show that the stream function for small disturbances to the state of rest obeys

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \beta\frac{\partial \psi}{\partial x} = 0 \quad ,$$

where β is a constant that should be found.

Obtain the dispersion relationship for plane-wave solutions of the form $\psi \propto \exp[i(kx + ly - \omega t)]$. Hence calculate the group velocity.

Show that if $\beta > 0$ then the phase of these waves always propagates to the left (negative x direction) but that the energy may propagate to either left or right.

20F Numerical Analysis

(i) Let A be an $n\times n$ symmetric real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$, where $\|\mathbf{v}_l\| = 1$. Given $\mathbf{x}^{(0)} \in \mathbb{R}^n$, $\|\mathbf{x}^{(0)}\| = 1$, the sequence $\mathbf{x}^{(k)}$ is generated in the following manner. We set

$$\mu = \mathbf{x}^{(k)\,T} A \mathbf{x}^{(k)},$$

$$\mathbf{y} = (A - \mu I)^{-1} \mathbf{x}^{(k)},$$

$$\mathbf{x}^{(k+1)} = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

Show that if

$$\mathbf{x}^{(k)} = c^{-1} \left(\mathbf{v}_1 + \alpha \sum_{l=2}^{n} d_l \mathbf{v}_l \right),$$

where α is a real scalar and c is chosen so that $\|\mathbf{x}^{(k)}\| = 1$, then

$$\mu = c^{-2} \left(\lambda_1 + \alpha^2 \sum_{j=2}^{n} \lambda_j d_j^2 \right).$$

Give an explicit expression for c.

(ii) Use the above result to prove that, if $|\alpha|$ is small,

$$\mathbf{x}^{(k+1)} = \tilde{c}^{-1} \left(\mathbf{v}_1 + \alpha^3 \sum_{l=2}^{n} \tilde{d}_l \mathbf{v}_l \right) + O(\alpha^4)$$

and obtain the numbers $\tilde{c}$ and $\tilde{d}_2, \ldots, \tilde{d}_n$.