UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Thursday 7 June 2001 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***2C, 5C*** *should be in one bundle and* ***11D, 14D*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1A Combinatorics

Write an essay on extremal graph theory. You should give proofs of at least two major theorems and you should also include a description of alternative proofs or of further results.

2C Representation Theory

Let G be the Heisenberg group of order p^3. This is the subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, x \in \mathbf{F}_p \right\}$$

of 3×3 matrices over the finite field $\mathbf{F}_p$ (p prime). Let H be the subgroup of G of such matrices with $a = 0$.

(i) Find all one dimensional representations of G.

[*You may assume without proof that* $[G, G]$ *is equal to the set of matrices in* G *with* $a = b = 0$.]

(ii) Let $\psi : \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{C}^*$ be a non-trivial one dimensional representation of $\mathbf{F}_p$, and define a one dimensional representation ρ of H by

$$\rho \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that $V_\psi = \operatorname{Ind}_H^G(\rho)$ is irreducible.

(iii) List all the irreducible representations of G and explain why your list is complete.

3C Galois Theory

Define the concept of separability and normality for algebraic field extensions. Suppose $K = k(\alpha)$ is a simple algebraic extension of k, and that $\mathrm{Aut}(K/k)$ denotes the group of k-automorphisms of K. Prove that
$|\mathrm{Aut}(K/k)| \leqslant [K : k]$, with equality if and only if K/k is normal and separable.

[*You may assume that the splitting field of a separable polynomial $f \in k[X]$ is normal and separable over k.*]

Suppose now that G is a finite group of automorphisms of a field F, and $E = F^G$ is the fixed subfield. Prove:

(i) F/E is separable.

(ii) $G = \mathrm{Aut}(F/E)$ and $[F : E] = |G|$.

(iii) F/E is normal.

[*The Primitive Element Theorem for finite separable extensions may be used without proof.*]

4B Differentiable Manifolds

Describe the Mayer-Vietoris exact sequence for forms on a manifold M and show how to derive from it the Mayer-Vietoris exact sequence for the de Rham cohomology.

Calculate $H^*(\mathbb{RP}^n)$.

5C Algebraic Topology

Write an essay on the definition of simplicial homology groups. The essay should include a discussion of orientations, of the action of a simplicial map and a proof of $\partial^2 = 0$.

6B Number Fields

For a prime number $p > 2$, set $\zeta = e^{2\pi i/p}, K = \mathbf{Q}(\zeta)$ and $K^+ = \mathbf{Q}(\zeta + \zeta^{-1})$.

(a) Show that the (normalized) minimal polynomial of $\zeta - 1$ over $\mathbf{Q}$ is equal to

$$f(x) = \frac{(x+1)^p - 1}{x}.$$

(b) Determine the degrees $[K : \mathbf{Q}]$ and $[K^+ : \mathbf{Q}]$.

(c) Show that

$$\prod_{j=1}^{p-1}(1 - \zeta^j) = p.$$

(d) Show that $\operatorname{disc}(f) = (-1)^{\frac{p-1}{2}} p^{p-2}$.

(e) Show that K contains $\mathbf{Q}(\sqrt{p^*})$, where $p^* = (-1)^{\frac{p-1}{2}} p$.

(f) If $j, k \in \mathbf{Z}$ are not divisible by p, show that $\frac{1-\zeta^j}{1-\zeta^k}$ lies in $\mathcal{O}_K^*$.

(g) Show that the ideal $(p) = p\mathcal{O}_K$ is equal to $(1 - \zeta)^{p-1}$.

7A Hilbert Spaces

Write an essay on the use of Hermite functions in the elementary theory of the Fourier transform on $L^2(\mathbb{R})$.

[*You should assume, without proof, any results that you need concerning the approximation of functions by Hermite functions.*]

8B Riemann Surfaces

Let λ and μ be fixed, non-zero complex numbers, with $\lambda/\mu \notin \mathbb{R}$, and let $\Lambda = \mathbb{Z}\mu + \mathbb{Z}\lambda$ be the lattice they generate in $\mathbb{C}$. The series

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}\Big[\frac{1}{(z - m\lambda - n\mu)^2} - \frac{1}{(m\lambda + n\mu)^2}\Big],$$

with the sum taken over all pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ other than (0,0), is known to converge to an *elliptic function*, meaning a meromorphic function $\wp : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ satisfying $\wp(z) = \wp(z + \mu) = \wp(z + \lambda)$ for all $z \in \mathbb{C}$. ($\wp$ is called the *Weierstrass function.*)

(a) Find the three zeros of $\wp'$ modulo Λ, explaining why there are no others.

(b) Show that, for any number $a \in \mathbb{C}$, other than the three values $\wp(\lambda/2), \wp(\mu/2)$ and $\wp((\lambda + \mu)/2)$, the equation $\wp(z) = a$ has exactly two solutions, modulo Λ; whereas, for each of the specified values, it has a single solution.

[*In (a) and (b), you may use, without proof, any known results about valencies and degrees of holomorphic maps between compact Riemann surfaces, provided you state them correctly.*]

(c) Prove that every *even elliptic function* $\phi(z)$ is a rational function of $\wp(z)$; that is, there exists a rational function R for which $\phi(z) = R(\wp(z))$.

9B Algebraic Curves

Write an essay on curves of genus one (over an algebraically closed field k of characteristic zero). Legendre's normal form should not be discussed.

10B Logic, Computation and Set Theory

What is a wellfounded relation, and how does wellfoundedness underpin wellfounded induction?

A formula $\phi(x, y)$ with two free variables *defines an* $\in$*-automorphism* if for all x there is a unique y, the function f, defined by $y = f(x)$ if and only if $\phi(x, y)$, is a permutation of the universe, and we have $(\forall xy)(x \in y \leftrightarrow f(x) \in f(y))$.

Use wellfounded induction over $\in$ to prove that all formulæ defining $\in$-automorphisms are equivalent to $x = y$.

11D Probability and Measure

State the first and second Borel-Cantelli Lemmas and the Kolmogorov 0-1 law.

Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random variables with distribution given by

$$\mathbb{P}(X_n = n) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0),$$

and set $S_n = \sum_{i=1}^n X_i$.

(a) Show that there exist constants $0 \leqslant c_1 \leqslant c_2 \leqslant \infty$ such that $\liminf_n (S_n/n) = c_1$, almost surely and $\limsup_n (S_n/n) = c_2$ almost surely.

(b) Let $Y_k = \sum_{i=k+1}^{2k} X_i$ and $\tilde{Y}_k = \sum_{i=1}^k Z_i^{(k)}$, where $(Z_i^{(k)})_{i=1}^k$ are independent with

$$\mathbb{P}(Z_i^{(k)} = k) = \frac{1}{2k} = 1 - \mathbb{P}(Z_i^{(k)} = 0), \quad 1 \leqslant i \leqslant k,$$

and suppose that $\alpha \in \mathbb{Z}^+$.

Use the fact that $\mathbb{P}(Y_k \geqslant \alpha k) \geqslant \mathbb{P}(\tilde{Y}_k \geqslant \alpha k)$ to show that there exists $p_\alpha > 0$ such that $\mathbb{P}(Y_k \geqslant \alpha k) \geqslant p_\alpha$ for all sufficiently large k.

[*You may use the Poisson approximation to the binomial distribution without proof.*]

By considering a suitable subsequence of (Y_k), or otherwise, show that $c_2 = \infty$.

(c) Show that $c_1 \leqslant 1$. Consider an appropriately chosen sequence of random times T_i, with $2T_i \leqslant T_{i+1}$, for which $(S_{T_i}/T_i) \leqslant 3c_1/2$. Using the fact that the random variables (Y_{T_i}) are independent, and by considering the events $\{Y_{T_i} = 0\}$, or otherwise, show that $c_1 = 0$.

12D Applied Probability

Define a *renewal process* and a *renewal reward process.*

State and prove the strong law of large numbers for these processes.

[*You may assume the strong law of large numbers for independent, identically-distributed random variables.*]

State and prove Little's formula.

Customers arrive according to a Poisson process with rate ν at a single server, but a restricted waiting room causes those who arrive when n customers are already present to be lost. Accepted customers have service times which are independent and identically-distributed with mean α and independent of the arrival process. Let P_j be the equilibrium probability that an arriving customer finds j customers already present.

Using Little's formula, or otherwise, determine a relationship between P_0, P_n, ν and α.

13E Information Theory

State the Kraft inequality. Prove that it gives a necessary and sufficient condition for the existence of a prefix-free code with given codeword lengths.

14D Optimization and Control

Consider the scalar system with plant equation $x_{t+1} = x_t + u_t$, $t = 0, 1, \ldots$ and cost

$$C_s(x_0, u_0, u_1, \ldots) = \sum_{t=0}^{s} \left[u_t^2 + \frac{4}{3} x_t^2 \right] .$$

Show from first principles that $\min_{u_0,u_1,\ldots} C_s = V_s x_0^2$, where $V_0 = 4/3$ and for $s = 0, 1, \ldots$,

$$V_{s+1} = 4/3 + V_s/(1 + V_s) .$$

Show that $V_s \to 2$ as $s \to \infty$.

Prove that C_∞ is minimized by the stationary control, $u_t = -2x_t/3$ for all t.

Consider the stationary policy π_0 that has $u_t = -x_t$ for all t. What is the value of C_∞ under this policy?

Consider the following algorithm, in which steps 1 and 2 are repeated as many times as desired.

1. For a given stationary policy π_n, for which $u_t = k_n x_t$ for all t, determine the value of C_∞ under this policy as $V^{\pi_n} x_0^2$ by solving for V^{π_n} in

$$V^{\pi_n} = k_n^2 + 4/3 + (1 + k_n)^2 V^{\pi_n} .$$

2. Now find k_{n+1} as the minimizer of

$$k_{n+1}^2 + 4/3 + (1 + k_{n+1})^2 V^{\pi_n}$$

and define π_{n+1} as the policy for which $u_t = k_{n+1} x_t$ for all t.

Explain why π_{n+1} is guaranteed to be a better policy than π_n.

Let π_0 be the stationary policy with $u_t = -x_t$. Determine π_1 and verify that it minimizes C_∞ to within 0.2% of its optimum.

15E Principles of Statistics

Write an account, with appropriate examples, of **one** of the following:

(a) Inference in multi-parameter exponential families;

(b) Asymptotic properties of maximum-likelihood estimators and their use in hypothesis testing;

(c) Bootstrap inference.

16D Stochastic Financial Models

Write an essay on the mean-variance approach to portfolio selection in a one-period model. Your essay should contrast the solution in the case when all the assets are risky with that for the case when there is a riskless asset.

17K Dynamical Systems

Define the rotation number $\rho(f)$ of an orientation-preserving circle map f and the rotation number $\rho(F)$ of a lift F of f. Prove that $\rho(f)$ and $\rho(F)$ are well-defined. Prove also that $\rho(F)$ is a continuous function of F.

State without proof the main consequence of $\rho(f)$ being rational.

18A Partial Differential Equations

Write an essay on **one** of the following two topics:

(a) The notion of *well-posedness* for initial and boundary value problems for differential equations. Your answer should include a definition and give examples and state precise theorems for some specific problems.

(b) The concepts of *distribution* and *tempered distribution* and their use in the study of partial differential equations.

19L Methods of Mathematical Physics

Show that $\int_0^\pi \mathrm{e}^{ix\cos t}\,dt$ satisfies the differential equation

$$xy'' + y' + xy = 0,$$

and find a second solution, in the form of an integral, for $x > 0$.

Show, by finding the asymptotic behaviour as $x \to +\infty$, that your two solutions are linearly independent.

20K Numerical Analysis

Write an essay on the computation of eigenvalues and eigenvectors of matrices.

21F Electrodynamics

The Liénard-Wiechert potential for a particle of charge q, assumed to be moving non-relativistically along the trajectory $y^\mu(\tau)$, τ being the proper time along the trajectory, is

$$A^\mu(x,t) \;=\; \frac{\mu_0 q}{4\pi}\,\frac{dy^\mu/d\tau}{(x-y(\tau))_\nu dy^\nu/d\tau}\Big|_{\tau=\tau_0}.$$

Explain how to calculate τ_0 given $x^\mu = (x,t)$ and $y^\mu = (y,t')$.

Derive Larmor's formula for the rate at which electromagnetic energy is radiated from a particle of charge q undergoing an acceleration a.

Suppose that one considers the classical non-relativistic hydrogen atom with an electron of mass m and charge $-e$ orbiting a fixed proton of charge $+e$, in a circular orbit of radius r_0. What is the total energy of the electron? As the electron is accelerated towards the proton it will radiate, thereby losing energy and causing the orbit to decay. Derive a formula for the lifetime of the orbit.

22F Foundations of Quantum Mechanics

(i) The two states of a spin-$\frac{1}{2}$ particle corresponding to spin pointing along the z axis are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. Explain why the states

$$|\uparrow,\theta\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle, \qquad |\downarrow,\theta\rangle = -\sin\frac{\theta}{2}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle$$

correspond to the spins being aligned along a direction at an angle θ to the z direction.

The spin-0 state of two spin-$\frac{1}{2}$ particles is

$$|0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\Big).$$

Show that this is independent of the direction chosen to define $|\uparrow\rangle_{1,2}$, $|\downarrow\rangle_{1,2}$. If the spin of particle 1 along some direction is measured to be $\frac{1}{2}\hbar$ show that the spin of particle 2 along the same direction is determined, giving its value.

[*The Pauli matrices are given by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad]$$

(ii) Starting from the commutation relation for angular momentum in the form

$$[J_3, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_3,$$

obtain the possible values of j, m, where $m\hbar$ are the eigenvalues of J_3 and $j(j+1)\hbar^2$ are the eigenvalues of $\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$. Show that the corresponding normalized eigenvectors, $|j,m\rangle$, satisfy

$$J_\pm|j,m\rangle = \hbar\left((j\mp m)(j\pm m+1)\right)^{1/2}|j,m\pm1\rangle,$$

and that

$$\frac{1}{n!}J_-^{\,n}|j,j\rangle = \hbar^n\left(\frac{(2j)!}{n!(2j-n)!}\right)^{1/2}|j,j-n\rangle, \qquad n \le 2j.$$

The state $|w\rangle$ is defined by

$$|w\rangle = e^{wJ_-/\hbar}|j,j\rangle,$$

for any complex w. By expanding the exponential show that $\langle w|w\rangle = (1+|w|^2)^{2j}$. Verify that

$$e^{-wJ_-/\hbar}J_3\,e^{wJ_-/\hbar} = J_3 - wJ_-,$$

and hence show that

$$J_3|w\rangle = \hbar\left(j - w\frac{\partial}{\partial w}\right)|w\rangle.$$

If $H = \alpha J_3$ verify that $|e^{i\alpha t}\rangle\, e^{-ij\alpha t}$ is a solution of the time-dependent Schrödinger equation.

23F Statistical Physics

Given that the free energy F can be written in terms of the partition function Z as $F = -kT\log Z$ show that the entropy S and internal energy E are given by

$$S = k\,\frac{\partial(T\log Z)}{\partial T}\,, \qquad E = kT^2\,\frac{\partial \log Z}{\partial T}\,.$$

A system of particles has Hamiltonian $H(\mathbf{p},\mathbf{q})$ where $\mathbf{p}$ is the set of particle momenta and $\mathbf{q}$ the set of particle coordinates. Write down the expression for the classical partition function Z_C for this system in equilibrium at temperature T. In a particular case H is given by

$$H(\mathbf{p},\mathbf{q}) = p_\alpha A_{\alpha\beta}(\mathbf{q})p_\beta + q_\alpha B_{\alpha\beta}(\mathbf{q})q_\beta\,.$$

Let H be a homogeneous function in all the $p_\alpha,\ 1 \le \alpha \le N$, and in a subset of the $q_\alpha,\ 1 \le \alpha \le M$ $(M \le N)$. Derive the principle of equipartition for this system.

A system consists of N weakly interacting harmonic oscillators each with Hamiltonian

$$h(p,q) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2\,.$$

Using equipartition calculate the classical specific heat of the system, $C_C(T)$. Also calculate the classical entropy $S_C(T)$.

Write down the expression for the quantum partition function of the system and derive expressions for the specific heat $C(T)$ and the entropy $S(T)$ in terms of N and the parameter $\theta = \hbar\omega/kT$. Show for $\theta \ll 1$ that

$$C(T) = C_C(T) + O(\theta)\,, \qquad S(T) = S_C(T) + S_0 + O(\theta)\,,$$

where S_0 should be calculated. Comment briefly on the physical significance of the constant S_0 and why it is non-zero.

24J Applications of Quantum Mechanics

Derive the Bloch form of the wave function $\psi(x)$ of an electron moving in a one-dimensional crystal lattice.

The potential in such an N-atom lattice is modelled by

$$V(x) = \sum_n \Big(-\frac{\hbar^2 U}{2m} \delta(x - nL) \Big).$$

Assuming that $\psi(x)$ is continuous across each lattice site, and applying periodic boundary conditions, derive an equation for the allowed electron energy levels. Show that for suitable values of UL they have a band structure, and calculate the number of levels in each band when $UL > 2$. Verify that when $UL \gg 1$ the levels are very close to those corresponding to a solitary atom.

Describe briefly how the band structure in a real 3-dimensional crystal differs from that of this simple model.

25J General Relativity

Discuss how Einstein's theory of gravitation reduces to Newton's in the limit of weak fields. Your answer should include discussion of:

(a) the field equations;

(b) the motion of a point particle;

(c) the motion of a pressureless fluid.

[*The metric in a weak gravitational field, with Newtonian potential ϕ, may be taken as*

$$ds^2 = dx^2 + dy^2 + dz^2 - (1 + 2\phi)dt^2.$$

The Riemann tensor is

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc}.\Big]$$

26H Fluid Dynamics II

Starting from the steady planar vorticity equation

$$\mathbf{u}\,.\nabla\omega = \nu\nabla^2\omega,$$

outline briefly the derivation of the boundary layer equation

$$uu_x + vu_y = UdU/dx + \nu u_{yy},$$

explaining the significance of the symbols used.

Viscous fluid occupies the region $y > 0$ with rigid stationary walls along $y = 0$ for $x > 0$ and $x < 0$. There is a line sink at the origin of strength πQ, $Q > 0$, with $Q/\nu \gg 1$. Assuming that vorticity is confined to boundary layers along the rigid walls:

(a) Find the flow outside the boundary layers.

(b) Explain why the boundary layer thickness δ along the wall $x > 0$ is proportional to x, and deduce that

$$\delta = \left(\frac{\nu}{Q}\right)^{\frac{1}{2}} x\ .$$

(c) Show that the boundary layer equation admits a solution having stream function

$$\psi = (\nu Q)^{1/2} f(\eta) \quad \text{with} \quad \eta = y/\delta\ .$$

Find the equation and boundary conditions satisfied by f .

(d) Verify that a solution is

$$f' = \frac{6}{1+\cosh(\eta\sqrt{2}+c)} - 1,$$

provided that c has one of two values to be determined. Should the positive or negative value be chosen?

27L Waves in Fluid and Solid Media

Derive the ray-tracing equations governing the evolution of a wave packet $\phi(\mathbf{x},t) = A(\mathbf{x},t)\exp\{i\psi(\mathbf{x},t)\}$ in a slowly varying medium, stating the conditions under which the equations are valid.

Consider now a stationary obstacle in a steadily moving homogeneous two-dimensional medium which has the dispersion relation

$$\omega(k_1,k_2) = \alpha\left(k_1^2+k_2^2\right)^{1/4} - Vk_1,$$

where $(V,0)$ is the velocity of the medium. The obstacle generates a steady wave system. Writing $(k_1,k_2) = \kappa(\cos\phi,\sin\phi)$, show that the wave satisfies

$$\kappa = \frac{\alpha^2}{V^2\cos^2\phi}.$$

Show that the group velocity of these waves can be expressed as

$$\mathbf{c}_g = V(\tfrac{1}{2}\cos^2\phi - 1, \tfrac{1}{2}\cos\phi\sin\phi).$$

Deduce that the waves occupy a wedge of semi-angle $\sin^{-1}\frac{1}{3}$ about the negative x_1-axis.